Lecture 3: The Navier-Stokes Equations:
Topological aspects

September 9, 2015

1 Goal

Topology is the branch of math which studies shape-changing objects; objects which can transform one into another without discontinuity (smooth mapping) are topologically equivalent. Topological invariants allow economic representations of the flow (less degrees of freedom). More practically, they are typically related to rotational degrees of freedom (motion without displacement).
2 Differential forms of fluid dynamics

Deformation (or Strain-rate): $D_{ij} \equiv \partial_i u_j$.

It measures the deformation of a volume of fluid due to fluid motion itself (kinematic distortion). Dimensionally, it is an inverse time-scale, rate of deformation.

Splits into two distinct components:

Shear-rate (symmetric) : $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

Vorticity (anti-symmetric) : $\Omega_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i)$

It is readily checked that the rotation tensor is shape-preserving, whereas the shear tensor changes the angles (non-conformal transformations). The Omega tensor is traceless by definition, the trace of the shear is the divergence of the flow velocity. As a result, strain describes dissipation while rotation/vorticity describes dissipation-free motion: Turning around is a good strategy to avoid dissipation!

3 Some typical flows

Radial flow, Vortex flow, Rigid flow, elongational flow.

4 Kelvin theorem

In inviscid flows the circulation of the velocity around a closed contour is conserved along the flow (topological invariant). This is a powerful result for the loop formulation of hydrodynamics (vortex methods). Circulation:

$$\Lambda = \oint_\gamma \vec{\omega} \cdot d\vec{l}$$

Kelvin theorem

$$\frac{DA}{Dt} = 0$$

5 Velocity-Vorticity representation

Vorticity is particularly relevant for high-Reynolds incompressible flows. Hence, it proves convenient to move from the primitive velocity-pressure (VP) representation to the auxiliary velocity-vorticity (VV) representation.

The main advantage is pressure elimination.

Note that vorticity is not just advected by the flow, it can be produced or consumed by coupling to the shear tensor:

$$\vec{\omega} \cdot \vec{S}$$

This term is very important, it drives vortex stretching and energy/entrophy transfer from large scale structures (thick tubes) to small ones (thin tubes).
However the relation between vorticity and velocity is non-local: the vorticity is a "vector" charge of the velocity: the inverse curl is the Green function, decays like $1/r$ in $d=3$, strong analogy with electrostatics. Just apply gauss theorem:

$$2\pi rv = \pi r^2 \omega = \Gamma$$

whence $v = \Gamma / 2\pi r$, where $\Gamma$ is the conserved angular momentum.

5.1 Potential flow is irrotational

Potential flows: $\vec{v} = -\nabla \Phi$, whence $\nabla \times \vec{v} = 0$.

Potential flow is a good approximation away from solid boundaries, where dissipation is nearly zero.

6 Enstrophy, Helicity

Enstrophy is a measure of rotational energy. Helicity measures the degree of swirling (spiral structures). Both are relevant to high Reynolds flows in 3d.

7 Two-dimensional flows

The vorticity-streamline representation is particularly insightful: elegant and computationally efficient:

$$u = \partial_y \psi, \ v = \partial_x \psi$$

is automatically zero divergence.

The streamlines are orthogonal to flux lines, which makes excellent "go-with-the-flow" coordinates.

8 Two-dimensional turbulence

2d fluid dynamics bears a direct link with Hamiltonian particle dynamics and describes coherent structures (vortices).

The vorticity equation takes a very elegant form

$$\partial_t \omega + J(\omega, \psi) = \nu \Delta \omega$$

where the Jacobian encodes the symplectic structure of hamiltonian dynamics. This is supplemented with the constraint

$$\Delta \psi = \omega$$

Vortices are long-lived metastable structures which almost completely escape dissipation. Dissipation lives on vortex boundaries. In a vortex structuer, vorticity and streamfunction are locally correlated, they achieve non-linear depletion, i.e. $J(\omega, \psi) = 0$. 
In 2d the vortext stretching is zero, hence vorticity and its powers (casimir invariants) are conserved. Enstrophy is conserved: inverse cascade Since enstrophy is conserved, the enstrophy flux is scale invariant, hence \( \frac{\delta v}{l} / (l/v) = \text{const} \), whence \( v \propto l \), the flow is regular. 2d flows in non cartesian geometry (spheres) are very relevant to meteorological fluid dynamics.

9 Summary

Rotational degrees of freedom play a major role in the physics of fluids, as they offer a strategy to minimize dissipation. The latter, described by the coupling of stress and strain tensor, cannot avoided, but takes place mostly in thin layers near solid walls (boundary layers).

Rotational degrees of freedom behave very differently in 2d and 3d. In the former case, rotational energy (enstrophy) is conserved (in the inviscid limit) and this gives rise to persistent coherent structures (vortices) with quasiparticle behavior. In 3d, vorticity can be either produced or destroyed, via vortex stretching, independently of viscosity. Vortex stretching is a fundamental mechanism in the development of 3d helical structures (spirals).
Topological Fluid Dynamics

TFD: how shape changes under the fluid flow.

Topological invariants:
Quantities that stay the same under flow-driven shape changes.

\[ D_t I = 0 \quad I = I(\vec{r}; t) \]

Topological invariants, why do they matter?
Topological invariants are “natural” degrees of freedom: they collect the essential info on the flow dynamics, hence disclose economic mathematical representations of the physics of fluids.
Deformation/Strain rate/Rotation

Deformation tensor: \( D_{ij} \equiv \partial_i u_j \)

\[
D_x = \frac{1}{2} (\partial_i u_j + \partial_j u_i) + \frac{1}{2} (\partial_i u_j - \partial_j u_i)
\]

\[
S_x = \frac{1}{2} (\partial_i u_j + \partial_j u_i)
\]

\[
\Omega_x = \frac{1}{2} (\partial_i u_j - \partial_j u_i)
\]

\[
\text{Tr}(S_x) = \partial_i u_i = \text{div} \, \bar{u}
\]

\[
\text{Tr}(\Omega_x) = 0
\]

Local flow topology

\[ D_x u_j = \lambda u_j \]

\( \lambda > 0 \) Expand \quad \lambda < 0 \) Compress \quad \text{div} \, \bar{u} = \lambda_1 + \lambda_2 + \lambda_3

*These are inverse time scales/normal bootstrap frequencies*

The deformation tensor governs/encodes the local flow topology

Strain goes with dissipation, Rotation is dissipation-free

---

Deformation: kinematics

Deformation \( D_{ij} \equiv \partial_i u_j \)

\[
x_i(t + dt) = x_i(t) + u_i(x(t)) \ast dt
\]

\[
\delta x_i(t + dt) = \delta x_i(t) + \frac{\partial u_i(x(t))}{\partial x_j} \delta x_j(t) \ast dt
\]

\[
\delta x_i(t + dt) = (\delta_i + D_{ij} \ast dt) \delta x_j(t)
\]

\[
V(t + dt) = \text{Det}[I + D_{ij}] \ast V(t)
\]

\[
\text{Det}[I + D_{ij}] = 1 + \text{div} \, \bar{u} \ast dt + O(dt^2)
\]
Differential forms

\[ \vec{\Omega} = \frac{1}{2} \begin{pmatrix} 0; \partial_x v - \partial_y u, \partial_y w - \partial_z u, \\ \partial_y u + \partial_z v; 0; \partial_z w + \partial_z v, \\ \partial_z u + \partial_z w; \partial_z v + \partial_z w; 0 \end{pmatrix} \]

\[ \vec{\omega} = \frac{1}{2} \begin{pmatrix} 0 & \omega_z & \omega_y \\ -\omega_z & 0 & \omega_x \\ -\omega_y & -\omega_x & 0 \end{pmatrix} \]
Deformation/Strain rate/Rotation

\[ S_v = \frac{1}{2} (\partial \mu_v + \partial \mu_v) \]
Strain rate: shear dissipation

\[ Tr(S_v) = \dot{\partial} \mu_v = \text{div} \dot{a} \]
Compression: bulk dissipation

\[ \Omega_v = \frac{1}{2} (\partial \mu_v - \partial \mu_v) \]
Rotation: No Dissipation

\[ \Omega_v = \omega_z; \Omega_v = \omega_z; \Omega_v = \omega_z; \]
Vorticity = pseudovector

*These are inverse time scales=internal bootstrap frequencies*

Whiteboard Example:
Compute $S, D, \Omega$, div for

Couette,
Poiseuille,
Rigid rotation,
Irrotational vortex,
Elongational (torture) flow
**Radial Flow**

\[ u(x, y) = kx \cdot f(r) \]
\[ v(x, y) = ky \cdot f(r) \]

\[ f(r) = 1 \]

K = signed "charge"

Compressible, Irrotational

---

**Rigid Rotation Flow**

\[ u = -\Omega y \]
\[ v = +\Omega x \]

Vorticity=2*Omega

Incompressible, Rotational
**Elongational (Torture) Flow**

\[ U = +\epsilon x \]
\[ V = -\epsilon y \]

Incompressible, Irrotational, Dilatant along \( x \), Shrinks along \( y \)

Volume stays the same

---

**Vortex Flow**

\[ U = -\Omega y \times f(r) \]
\[ V = +\Omega x \times f(r) \]

Rotation=Omega\*f(r)

Divergence = 0

Vorticity = -2*Omega*1/f - Omega*df/dr if \( f(r)=\frac{1}{r^2} \),

No vorticity

Rotation without vorticity!
Velocity-Vorticity

\[ \{\rho, \vec{u}, p\} \rightarrow \{\vec{u}, p\} \rightarrow \{\vec{u}, \vec{\omega}\} \]

\[ \vec{\omega} = \nabla \times \vec{u} \]

Degree of local rotation

**Eliminates pressure**

Useful for nearly-inviscid flows
Potential Flow

\[ \ddot{u} = -\nabla \Phi \implies \ddot{\omega} = 0 \]

Potential \(\rightarrow\) Irrotational

Potential & Incompressible

\[ \Delta \Phi = 0 \quad \Phi = \text{const} \]

Analytic function: very useful for 2D incompressible hydrodynamics

Rotational/Irrotational

\[ \ddot{\omega} \equiv \nabla \times \ddot{u} = 0 \]
**Turbo-jungle vorticity**

Low viscous, incompressible flows are vorticity-rich.

---

**Kelvin theorem**

Circulation: \( \Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l} \)

Gauss theorem: \( \oint_C \mathbf{u} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} \ dS = \omega A \)

Conservation of angular momentum:

\[ \omega_1 A_1 = \omega_2 A_2 \]

\[ \omega(r) \propto \frac{1}{r^2} \]
**Kelvin theorem: long range decay**

$$
\omega = \frac{\Gamma}{\pi r^2}
$$

$$
2\pi ru = \omega \pi r^2 = \Gamma
$$

$$
u_\perp(r) = \frac{\Gamma}{2\pi r} \quad \omega(r) = \frac{\Gamma}{\pi r^2}
$$

**Vortex stretching**

\[ \partial_t \omega + \bar{u} \cdot \nabla \omega = \bar{\omega} \cdot \nabla \bar{u} + \nu \Delta \bar{\omega} \quad \text{Pressure-free!} \]

\[ \nabla \times \bar{u} - \bar{\omega} = 0 \quad \text{Instantaneous constraint} \]

Vortex stretching:
Source or sink of vorticity
3D: Enstrophy-helicity

**Enstrophy**

Rotational energy density:

\[ \Omega = \frac{\vec{\omega} \cdot \vec{\omega}}{2} \]

Vorticity Stretch: Finite-time blow-up?

\[ D_t (\omega^2 / 2) = \vec{\omega} \cdot \nabla \vec{u} \cdot \vec{\omega} + \nu \Delta (\omega^2 / 2) \]

\[ D_t \Omega = s \Omega^{3/2} + \nu \Delta \Omega \]

\[ \Omega(t) \approx (t - t_{bu})^{-2} \]

Nonlinear depletion can block it off:

\[ \vec{\omega} \cdot \nabla \vec{u} = 0 \]
**Vortex methods**

Pressure-free!

\[ D_t \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u} + \nu \Delta \vec{\omega} \]

Collection of vortices - electromagnetic

\[ \vec{\omega} = \nabla \times \vec{u} \quad \vec{u} = \frac{\Gamma}{2\pi r} \hat{r} \quad \Gamma = \pi r^2 \omega \]

Long-range (electrosiatic) interactions

**Helicity**

\[ h = \vec{u} \cdot \vec{\omega} = u \parallel \omega \]

2d; \quad h = 0

3d animal

Swirl motion, Dynamo

Beltrami flows (1d spirals):

\[ \vec{u} = \lambda \vec{\omega} \]
2D is different!

Why?

2d: no vortex stretching!

\[ \vec{\omega} \cdot \nabla \vec{u} \equiv 0 \]

\[ \vec{u} = [u_x(x, y), u_y(x, y), 0] \]

\[ \vec{\omega} = (0, 0, \omega_z) \]

New family of strictly 2d invariants:
erestraphy and her high-order relatives (Casimirs)

\[ \frac{d}{dt} \left[ \int_{A} \omega \cdot \delta \cdot dy \right] = 0 \]
2d: Vorticity-Streamfunction

Streamlines: local tangent to the 2d flow:
\[
\vec{u} \cdot \nabla \psi = 0 \implies \begin{cases} u = -\partial_y \psi \\ v = \partial_x \psi \end{cases}
\]

Built-in incompressibility:
\[
div \vec{u} = \partial_x u + \partial_y v = 0
\]
\[
\omega = \nabla \times \vec{u} \cdot \hat{z} = \Delta \psi
\]

Potential $\to$ Irrotational: $\Delta \psi = 0$

Potential Flow: 2D

\[
\vec{u} = -\nabla \Phi
\]
\[
u = -\partial_x \Phi = -\partial_y \Psi
\]
\[
v = -\partial_y \Phi = \partial_x \Psi
\]
\[
(\partial_x \Phi)(\partial_y \Psi) + (\partial_y \Phi)(\partial_x \Psi) = vv + v(-\omega) = 0
\]
\[
\Delta \Phi = \Delta \Psi = 0
\]

Conformal mapping:
\[
z = (x + iy) \mapsto w = f(z) = \Phi + i\Psi
\]
Body-fitted coordinates

2d turbulence: Vorticity-Stream

Two-dimensional: spectral methods

\[ \partial_t \omega + J(\psi, \omega) = \nu \Delta \omega \]
\[ \Delta \psi = \omega \]

\[ J(\psi, \omega) \equiv \partial_x \psi \partial_y \omega - \partial_x \omega \partial_y \psi \]

Nonlinear depletion: coherent structures (vortices)

\[ J[\psi, \omega = \omega(\psi)] = 0 \]
**Ideal 2d: Hamiltonian**

Symplectic dynamics:

\[
\frac{dx}{dt} = [\psi, x] \quad \frac{dy}{dt} = [\psi, y]
\]

\[
\frac{dx}{dt} = \partial_y \psi \partial_x - \partial_x \psi \partial_y, x = \partial_x \psi = u
\]

\[
\frac{dy}{dt} = \partial_y \psi \partial_y - \partial_x \psi \partial_y, y = \partial_y \psi = v
\]

Borrow a lot from particle dynamics!
Hamiltonian streaming + vortex mergers/breakup

**Axys-symmetric coherent structures**

\[ \Delta \psi = \omega(\psi) \quad \psi(r) \]

\[ \omega(r) \]
**Coherent structures**

\[ \omega = \omega(\psi) \]

Non-linear depletion

\[ u \partial_x \omega + v \partial_y \omega = -\partial_x \psi \partial_x \omega + \partial_y \psi \partial_y \omega = J(\psi, \omega) \]

\[ \omega = \omega(\psi) \Rightarrow J = \omega [-\partial_y \psi \partial_x \psi + \partial_x \psi \partial_y \psi] = 0 \]

**Coherent structures**

\[ \omega = \omega(\psi) \]

Non-linear depletion

\[ J[\psi, \omega(\psi)] = 0 \]

Cascade blocking;
Long-lived metastable states

Enstrophy cascade:

\[ \frac{\omega^2(t)}{1/u(t)} = \text{const.} \quad \frac{u'(t)}{l^3} = \text{const.} \]

\[ u(l) = \text{const} \cdot l \quad \text{REGULAR!} \]
**2d turbulence: inverse cascade**

Cascade blocking;  
Long-lived metastable states

Enstrophy cascade:

\[
\frac{u^2(l)}{l / m(l)} = \text{const.}
\]

\[
u(l) = \text{const} \times l
\]

REGULAR!

Summary

Topological invariants highlight conserved quantities under flow;  
Economic description of the flow (less degrees of freedom)

Vorticity is very relevant to incompressible flows, high where  
dissipation is low (Kelvin theorem)

Vorticity equation in 3d contains a crucial vortex stretching term

Vortex stretching is identically zero in 2d, persistent structures (vortices)

2d flows are well suited to vorticity-streamfunction representation

2d turbulence is quasi-hamiltonian: coherent structures behave like  
metastable particles. Very relevant to meteorological fluid dynamics.
Assignements

1. Compute shear and vorticity for Poiseuille, Couette, Elongational flow

2. Show that 2d flows have no vortex stretching

2. Write a computer program to track the deformation of a square within a Couette flow