1 Goal

In this Lecture, we shall present the main ideas behind the simulation of fluid turbulence. We first discuss the case of the direct numerical simulation, in which all scales of motion within the grid resolution are simulated and then move on to turbulence modeling, where the effect of unresolved scales on the resolved ones is taken into account by various forms of modeling,
2 Fluid turbulence

Turbulence is the peculiar state of matter, typical of gases and liquids, characterized by the simultaneous interaction of a broad spectrum of scales of motion, both in space and time. Such simultaneous interaction gives rise to a host of morpho-dynamical complexity which makes the mid/long-term behavior of turbulent flows very hard to predict, weather forecasting possibly offering the most popular example in point.

The importance of turbulence, from both theoretical and practical points of view, cannot be overstated. Besides the intellectual challenge associated with the predictability of the dynamics of complex nonlinear systems, the practical relevance of turbulence is even more compelling as one thinks of the pervasive presence of fluids across most natural and industrial endeavours: air and blood flow in our body, gas flows in car engines, geophysical and cosmological flows, to name but a few. Even though the basic equations of motion of fluid turbulence, the Navier-Stokes equations, are known for nearly two centuries, the problem of predicting the behaviour of turbulent flows, even only in a statistical sense, is still open to this day.

In the last few decades, numerical simulation has played a leading role in advancing the frontier of knowledge of this over-resilient problem (often called the last unsolved problem of classical physics).

To appraise the potential and limitations of the numerical approach to fluid turbulence, it is instructive to revisit some basic facts about the physics of turbulent flows.

The degree of turbulence of a given flow is commonly expressed in terms of a single dimensionless parameter, the Reynolds number, defined as

\[ \text{Re} = \frac{UL}{\nu}, \]  

(1)

where \( U \) is a typical macroscopic flow speed, \( L \) the corresponding spatial scale and \( \nu \) is the kinematic molecular viscosity of the fluid. The Reynolds number measures the relative strength of advective over dissipative phenomena in a fluid flow: \( \text{Re} \sim u\nabla u/(\nu\Delta u) \). Given the fact that many fluids feature a viscosity around \( 10^{-6} \text{m}^2/\text{s} \) and many flows of practical interest work at speeds around and above \( U = 1 \text{ m/s} \) within devices sized around and above \( L = 1 \text{ m} \), it is readily checked that \( \text{Re} = 10^6 \) is commonplace in real life applications. In other words, non-linear inertia far exceeds dissipation, a hallmark of macroscopic phenomena.

The basic physics of turbulence is largely dictated by the way energy is transferred across scales of motion.

To date, turbulence energetics is best understood in terms of an energy cascade from large scales (\( l \sim L \)) where energy is fed into the system, down to small scales where dissipation takes central stage (see Fig. ??).

This cascade is driven by the nonlinear mode–mode coupling in momentum space associated with the advective term of the Navier–Stokes equations, \( \vec{u} \cdot \nabla \vec{u} \).
In real space, this is the steepening of ocean waves as they approach the shore, we are all familiar with.

The smallest scale reached by the energy cascade is known as the Kolmogorov length, $l_k$, and marks the the point where dissipation takes over advection and organized fluid motion dissolves into incoherent molecular motion (heat).

According to the celebrated Kolmogorov (1941) scaling theory, the Kolmogorov length can be estimated as:

$$ l_k \sim \frac{L}{Re^{3/4}}. $$

The estimate (2) follows straight from Kolmogorov’s assumption of a constant energy flux across all scales of motion.

Let $\epsilon(l) \sim \frac{\delta u^2(l)}{\tau(l)}$ the rate of change of the kinetic energy associated with a typical eddy of size $l$ (mass=1 for simplicity). By taking $\tau(l) \sim l/\delta u(l)$, we obtain

$$ \epsilon(l) \sim \frac{\delta u^3(l)}{l} $$

Kolmogorov’s assumption of scale invariance then implies:

$$ \delta u(l) = U \left( \frac{l}{L} \right)^{1/3} $$

where $U$ is the macroscopic velocity at the integral scale $L$. Note that this realtion implies that the velocity gradient $\delta u(l)/l$ goes to infinity in the limit $l \to 0$, indicating that the flow configuration is singular, i.e. non-differentiable.

By definition, the Kolmogorov (or dissipative) length is such that dissipation and inertia come to an exact balance, hence:

$$ \delta u(l_k) l_k / \nu = 1 $$

namely,

$$ U \left( \frac{l_k}{L} \right)^{1/3} (l_k / L) = \nu / L $$

whence the expression (2).

The corresponding energy spectrum $E(k) \equiv u^2(k), u(k)$ being the Fourier transform of $u(l)$, is readily computed to scale like

$$ E(k) = \text{const.} \ k^{-5/3}, \quad (3) $$

the famous Kolmogorov $-5/3$ spectrum.

The qualitative picture emerging from this analysis is fairly captivating: a turbulent flow in a cubic box of size $L$, at a given Reynolds number $Re$, is represented by a collection of $N_k \sim (L/l_k)^3$ Kolmogorov eddies.

By identifying each Kolmogorov eddy with an independent degree of freedom, a ”quantum of turbulence”, (sounds like a good title for the next Bond’s
we conclude that the number of degrees of freedom involved in a turbulent flow at Reynolds number $Re$ is given by:

$$N_{dof} \sim Re^{9/4}. \quad (4)$$

According to this estimate, even a standard flow with $Re = 10^6$, features more than $10^{13}$ degrees of freedom, enough saturate the most powerful present-day computers!

This sets the current bar of Direct Numerical Simulation (DNS) of turbulent flows, manifestly one falling short of meeting the needs raised by many real life applications.

The message comes down quite plain: *computers alone won’t do*!

Of course, this does not mean that computer simulation is useless. Quite the contrary, it plays a pivotal role as a complement and sometimes even an alternative to experimental studies.¹

Yet, the message is that sheer increase of raw compute power must be accompanied by a corresponding advance of computational methods.

Fluid turbulence is very sensitive to the space dimensionality. For instance, three-dimensional fluids support finite dissipation even in the (singular) limit of zero viscosity, while two-dimensional ones do not. This is rather intuitive, since three dimensional space offers much more morpho-dynamical freedom than two or one-dimensional ones, hence the flow can go correspondingly "wilder".

Therefore, we shall begin our discussion with two-dimensional turbulence.

2.1 Two-dimensional turbulence

As noted above, two-dimensional turbulence differs considerably from three-dimensional turbulence. In particular, two-dimensional turbulence supports an infinite number of (Casimir) invariants which can only exist in ‘Flatland’.

These read as follows:

$$\Omega_{2p} = \int_V |\omega|^{2p} V, \quad p = 1, 2, \ldots, \quad (5)$$

where

$$\vec{\omega} = \nabla \times \vec{u} \quad (6)$$

is the vorticity of the fluid occupying a region of volume $V$.

By taking the curl of the Navier–Stokes equations, one obtains:

$$D_t \vec{\omega} = \nu \Delta \vec{\omega} + \vec{\omega} \cdot \nabla \vec{u}. \quad (7)$$

where $D_t$ is the material derivative. It is easily seen that the second term on the right-hand side, acting as a source/sink of vorticity, is identically zero in two dimensions because the vorticity is orthogonal to the flow field.

As a result, in the inviscid limit $\nu \to 0$, vorticity is a conserved quantity (topological invariant), and so are all its powers. The fact that vorticity, and particularly enstrophy $\Omega_2$, is conserved has a profound impact on the scaling laws of 2D turbulence. It can be shown that the enstrophy cascade leads to a fast decaying ($k^{-3}$) energy spectrum, as opposed to the much slower $k^{-5/3}$ fall-off of 3D turbulence. This regularity derives from the existence of long lived metastable states, vortices, that manage to escape dissipation for quite long times. The dynamics of these long-lived vortices has been studied in depth by various groups and reveals a number of fascinating aspects whose description goes however beyond the scope of this Lecture.

3 Turbulence modeling

4 Sub-grid scale modeling

Most real-life flows of practical interest exhibit Reynolds numbers far too high to be amenable to direct simulation by present-day computers, and for many years to come.

This raises the challenge of predicting the behavior of highly turbulent flows without directly simulating all scales of motion but only those that fit the available computer resolution. The effect of the unresolved scales of motion on the resolved ones must therefore be modeled.

To this purpose, it proves expedient to split the actual velocity field into large-scale (resolved) and short-scale (unresolved) components:

$$u_a = U_a + \tilde{u}_a$$

and seek turbulent closures yielding the Reynolds stress tensor:

$$\sigma_{ab} = \rho \langle \tilde{u}_a \tilde{u}_b \rangle$$

in terms of resolved field $U_a$. Here, brackets denote ensemble-averaging, typically replaced by space-time averaging on the grid. This task makes the object of an intense area of turbulence research, known as turbulence modeling, or sub-grid scale (SGS) modeling.

One of the most powerful heuristics behind SGS is the concept of eddy viscosity. This idea, a significant contribution of kinetic theory to fluid turbulence, assumes that the effect of small scales on the large ones can be likened to a diffusive motion caused by random collisions. Small eddies are kinematically transported without distortion by the large ones, while the large ones experience diffusive Brownian-like motion due to erratic collisions with the small eddies.

One of the simplest and most popular models in this class is due to Smagorinski [?]. This model is based on the following representation of the Reynolds stress tensor:

$$\sigma_{ab} = \rho \nu_e |S| S_{ab},$$
where $S_{ab} = \left( \partial_a U_b + \partial_b U_a \right)/2$ is the large-scale strain tensor of the incompressible fluid and where $\nu_e$ is the effective eddy-viscosity, defined as the sum of the molecular plus the turbulent one:

$$\nu_e(S) = \nu_0 + \nu_t(S)$$

The specific expression of the Smagorinski eddy viscosity is as follows:

$$\nu_e = \nu_0 + C_S \Delta^2 |S|, \quad |S| = \left( \sum_{a,b} 2 S_{ab} S_{ab} \right)^{1/2},$$

where $C_S$ is an empirical constant of the order 0.1, and $\Delta$ is the mesh size of the numerical grid ($\Delta = 1$ in lattice units).

The stabilizing role of the effective viscosity is fairly transparent: whenever large strains develop, $\sigma_{ab}$ becomes large and turbulent fluctuations are damped down much more effectively than by mere molecular viscosity, a negligible effect at this point.

### 4.1 Two-equation models

The main virtue of the Smagorinski SGS model is simplicity: it is an algebraic model which does not imply any change in the mathematical structure of the Navier–Stokes equations. However, it is known to cause excessive damping near the walls, where $S$ is highest.

The next level of sophistication is to link the eddy viscosity to the actual turbulent kinetic energy:

$$k = \frac{1}{2} \langle \tilde{u}^2 \rangle$$

and dissipation

$$\epsilon = \frac{\nu}{2} \sum_{a,b} \langle (\partial_a \tilde{u}_b)^2 \rangle.$$  

These quantities are postulated to obey advection–diffusion–reaction equations of the form [?]:

$$\dot{D}_{\epsilon} k - \nu_k \Delta k = S_k,$$

$$\dot{D}_{\epsilon} \epsilon - \nu_\epsilon \Delta \epsilon = S_\epsilon,$$  

where $\nu_k$, $\nu_\epsilon$ are effective non-linear viscosities for $k$ and $\epsilon$, respectively, and $S_k$, $S_\epsilon$ are local sources of $k$ and $\epsilon$ representing the difference between production and dissipation [?]. Finally, $\dot{D}_t \equiv \partial_t + \tilde{u}_a \partial_a$ with $\tilde{u}_a \equiv u_a - \partial_a \nu$, where $\nu \equiv \nu_k$ or $\nu \equiv \nu_\epsilon$ [?]. These equations constitute the well-known $k$–$\epsilon$ model of turbulence, still one of the most popular options for industrial applications. However, the practical use of the $k$–$\epsilon$ equations may raise concerns of convergence, due to the non-linear dependence of the eddy viscosities and stiffness of the source terms as well.
4.2 Reynolds-Averaged Navier-Stokes (RANS)

The $k - \epsilon$ model often provides an improvement over Smagorinski, but it is still liable to criticism. In particular, it does not account for the higher directional nature of turbulence near solid boundaries. To cope with this problem, a further level of sophistication is introduced, by formulating dynamic equations for the Reynolds stress itself $\sigma_{ab}$. These equations tends to be rather cumbersome, especially in connection with the formulation of proper boundary conditions. Even though they enjoy significant popularity in the CFD engineering community, we shall not delve any further into this option.

5 Summary

The modeling of fluid turbulence still stands as one of the major open topics in modern science and engineering. Current computer capabilities allow the full simulation of flows up to Reynolds numbers of the order of $10^4$, which is short by several orders of magnitude for many engineering applications, let alone environmental and geophysical ones. To fill this gap, the current practice is extensive resort to various forms of turbulence modeling.
Transition to turbulence

Incompressible Turbulence

The Navier-Stokes equations are known for long, but very hard to solve:

\[ \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} = \nu \Delta \tilde{u} - \nabla p \]
\[ \rho = 1; \quad \nabla \cdot \tilde{u} = 0 \]

\[ \text{Re} = \frac{UL}{\nu} \quad \text{Re} \geq 10^6 \]
Energy transfer/dissipation: real space

Non-linear energy transfer (energy flux):
\[ \dot{\theta}(r,t) = \ddot{u} \cdot (\ddot{u} \cdot \nabla)\ddot{u} = \ddot{u} \cdot \nabla(u^2/2) + \ddot{u} \times \ddot{\omega} \]
\[ \ddot{u} \cdot \nabla\ddot{u} = \nabla(u^2/2) + \ddot{u} \times \ddot{\omega} \]
\[ \ddot{u} \cdot \nabla(u^2/2) = \nabla(\ddot{u}u^2/2) - (u^2/2)\nabla\cdot\ddot{u} \]
Locally nonzero, Globally zero: interscale transfer

Energy (density) dissipation rate \( \frac{dK}{dt} = \text{Dissipative Force} \cdot \text{Velocity} \):
\[ \dot{\varepsilon}(r,t) = u \partial_j \sigma_{ij} = -\sigma_{ij} \partial_j u_i = -[\mu S_j + \lambda(\partial_i u_i)\delta_{ij}]D_j \]
\[ (D_j = S_j + \Omega_j = S_j - \Omega_i) \]
\[ \dot{\varepsilon}(r,t) = -(\mu S_j)(S_j + \Omega_j) = -\mu S_j S_j = -\mu S^2 < 0 \]
Locally negative, Globally negative:
Momentum space: Fourier Transform

Discrete Fourier Transform (inverse):

\[ \vec{u}(\vec{r}', t) = \sum_{n=0}^{N} \vec{u}_n(t) e^{i \vec{k}_n \cdot \vec{r}' + c.c.} \]

\[ \vec{k}_n = k_{n_i} = \frac{2\pi}{L_i} n_i \quad i = x, y, z \]

Plus: Differential → Algebraic

\[ \nabla \leftrightarrow i \vec{k} \]
\[ \Delta \leftrightarrow -k^2 \]

Energy spectrum

Total kinetic energy:

\[ 2K = \int_V \vec{u} \cdot \vec{u} d\vec{r} = \int_V \left( \sum_{n=0}^{N} \vec{u}_n(t) e^{i \vec{k}_n \cdot \vec{r}} \right) \cdot \left( \sum_{n=0}^{N} \vec{u}_n(t) e^{i \vec{k}_n \cdot \vec{r}} \right) d\vec{r} \]

\[ = \sum_{n=0}^{N} \sum_{m=0}^{N} (\vec{u}_n \cdot \vec{u}_m) \int_V e^{i (\vec{k}_n + \vec{k}_m) \cdot \vec{r}} d\vec{r} \cdot \delta(\vec{k}_n + \vec{k}_m) \]

\[ = \sum_{n=0}^{N} (\vec{u}_n \cdot \vec{u}_n) = \sum_{n=0}^{N} |\vec{u}_n|^2 \]
Steep spectra = smooth signals and vice versa

\[ E(k) = \text{const} \quad \text{white noise} \]

\[ E(k) = \exp\left(-\left(k/k_0\right)^2\right) \]

\[ E(k) = E(k_0)\left[1+(k/k_0)^2\right]^{p/2} \]
**Kolmogorov 1941: energy cascade**

Energy supply: large scales

Energy transfer: from large to small

Viscous dissipation: Small scales

\[ \frac{1}{L} \ll k \ll \frac{1}{L_d} \]

**Navier Stokes: spectral formulation**

\[ \frac{\partial}{\partial t} \hat{u} + \hat{u} \cdot \nabla \hat{u} = -\nabla \hat{p} + \nabla \times \nabla \times \hat{A} - \hat{\nu} \Delta \hat{u} \]

\[ \hat{u}_i(x) \rightarrow \hat{u}_i(k) = \mathcal{F}[u_i] \]

\[ \hat{\Delta} \hat{u}_i \rightarrow -\hat{\nu} k^2 \hat{u}_i(k) \]

Dissipation dampes high-k short wavelengths

\[ \Delta \hat{p} = \text{div} \hat{A} \rightarrow -k^2 \hat{p}_i = i k \cdot \hat{\Lambda} \]

Pressure isotropizes the modes

Quadratic convolution: mode-mode coupling

Energy transfer: Complexity O(N^2)

\[ A_i = u \cdot \nabla \hat{u}_i \rightarrow \hat{\Lambda}_i(k) = \sum_k \hat{\nu} \hat{u}_i(k - k') \hat{\nu}(k') \]
Turbulent energy spectrum: broad and gapless!

Kolmogorov 1941: energy cascade

Energy supply:
Large scales

Energy spectrum:
No equipartition!

Energy dissipation:
Small scales

\[ \frac{l_d}{L} = Re^{-3/4} \]

\[ E(k) = E_0 (k / k_d)^{-5/3} \]

\[ 1/L \ll k \ll 1/l_d \]
**Velocity fluctuations**

\[ \epsilon(l) = \frac{\delta u^2(l)}{l / \delta u(l)} = \text{const.} \]

\[ \frac{\delta u^3(l)}{l} = \text{const.} \]

\[ \delta u(l) = \text{const.} \cdot l^{1/3} \]

\[ \frac{\delta u(l)}{l} = \text{const.} \cdot l^{-2/3} \quad \text{Singular!} \]

\[ \epsilon = \frac{dK}{dt} = -\nu \int V u^2 \, dV \rightarrow \epsilon_0 < 0 \quad \nu \rightarrow 0 \]

Dissipation is a die-hard even at vanishing viscosity!

**K41: energy spectrum**

\[ E(k) = u^2(k)k^2 = k^2 \int \delta u^2(l)e^{-\omega l^2} \, dl \]

\[ E(k) = k^{-(i+2)} \int \omega^{i} e^{-\omega} (kl)^2 \, d(kl) \propto \text{const} \cdot k^{-5/3} \]

**General rule:**

\[ \delta u(l) = l^p \Leftrightarrow E(k) = k^{-(i+2)} \]

- **Linear:** \( p=1 \) \( k^{1} \]
- **Diffusion:** \( p=3/2 \) \( k^{3/2} \]
- **Singular:** \( p=1 \) \( k^{1} \]

14
Kolmogorov=Dissipative length

- Kolmogorov length: \( \text{Re}(l_d) = 1 \)

\[
\frac{\delta u(l_d) \cdot l_d}{v} = \frac{U \cdot (l_d/L)^{1/3} \cdot l_d}{v} = 1
\]

\[
(UL)^{(U/L)^{1/3}} = 1
\]

\[
l_d = \frac{L}{\text{Re}^{3/4}}
\]

L=1m:
Re=10^4  \( l_d = 1 \)mm
Re=10^8  \( l_d = 1 \)micron

Turbulence: Degrees of freedom

- Kolmogorov length

\[
l_d = \frac{L}{\text{Re}^{3/4}}
\]

\[
\text{DOF} = (L/l_d)^3 = \text{Re}^{9/4} < \text{Re}^3
\]

\[
\text{Work} \geq (L/l_d)^4 = \text{Re}^3
\]

Faucet: Re=10^4, DOF=10^9, Work=10^12
Car : Re=10^6, DOF=10^14,
Geo : Re=10^9, DOF=10^20
Astro : Re=10^10, DOF=10^22, Work=10^30
Message

Computers alone won’t do!

Two-dimensional turbulence
Two-dimensional turbulence

Enstrophy cascade

More regular than 3d

2d turbulence: inverse cascade

Cascade blocking;
Long-lived metastable states

Enstrophy cascade:

\[ \frac{u^2(l)}{l^3} = \text{const.} \]

\[ u(l) = \text{const} \times l \]

REGULAR!

3d Energy cascade: SINGULAR!

\[ \frac{u^3(l)}{l} = \text{const.} \]
2d turbo: Vorticity-Streamfunction

Two-dimensional: spectral methods

\[ \partial_t \omega + J(\psi, \omega) = \nu \Delta \omega \]

\[ \Delta \psi = \omega \]

\[ J(\psi, \omega) = \partial_x \psi \partial_y \omega - \partial_x \omega \partial_y \psi \]

Nonlinear depletion: coherent structures (vortices)

\[ J[\psi, \omega = \omega(\psi)] = 0 \]

Ideal 2d: Hamiltonian

Symplectic dynamics:

\[ \frac{dx}{dt} = [\psi, x] \]

\[ \frac{dy}{dt} = [\psi, y] \]

\[ \frac{dx}{dt} = \partial_x \psi \partial_y \omega - \partial_x \omega \partial_y \psi = -\partial_y \psi = u \]

\[ \frac{dy}{dt} = \partial_x \psi \partial_y \omega - \partial_x \omega \partial_y \psi = \partial_y \psi = v \]

Borrow a lot from particle dynamics!
Hamiltonian streaming + vortex mergers/breakup
Axys-symmetric coherent structures

$$\Delta \psi = \omega(\psi) \quad \psi(r)$$

Coherent structures

$$\omega = \omega(\psi)$$

Non-linear depletion

$$u \partial_x \omega + v \partial_y \omega = -\partial_y \psi \partial_x \omega + \partial_x \psi \partial_y \omega = J(\psi, \omega)$$

$$\omega = \omega(\psi) \Rightarrow J = \omega \left[ -\partial_y \psi \partial_x \psi + \partial_x \psi \partial_y \psi \right] = 0$$
Coherent structures

\[ \omega = \omega(\psi) \]

Non-linear depletion

\[ J[\psi, \omega(\psi)] = 0 \]

Cascade blocking;
Long-lived metastable states

Enstrophy cascade:

\[ \frac{\omega^2(l)}{l/u(l)} = \text{const.} \quad \frac{u'(l)}{l^3} = \text{const.} \]

\[ u(l) = \text{const} \times l \] REGULAR!

Turbulence Modeling
Small and Large Eddies

Turbulence modeling

Effects of small (unresolved) scales on large (resolved) ones
Reynolds stress tensor

- Slow and fast DOF: \( \tilde{u} = \tilde{U} + u' \); \( \tilde{U} = \langle \tilde{u} \rangle \)

\[
< NS(\tilde{u}) > = NS(\tilde{U}) + \text{div} < \tilde{u}'\tilde{u}' >
\]

\[
\vec{F}_{\text{turbo}} = \nabla \cdot \tilde{T}, \quad \tilde{T} = < \tilde{u}'\tilde{u}' >
\]

"Little problem: pdf \( <...> \) is NOT known!

\[
\tilde{T} = \nu_{\text{turbo}} \nabla \tilde{U}
\]

Big BUT: No scale separation!

Turbulence: NO scale separation

Small eddies are swept away by large eddies
Large eddies experience random collisions from small ones
Brownian motion? NO! Advection/Diffusion is scale-dependent
Non-gaussian fluctuations, intermittency, bursts, rare events
Dissipative: No Hamiltonian, no standard statistical ensembles
### Turbulence Cost

\[ l_d = \frac{L}{Re^{3/4}} \]

**Memory** \( M \propto L^3 \propto Re^3 \)

**CPU** \( T \propto L^{4+} \propto Re^{4+} \)

\[ DOF = \left( \frac{L}{l_d} \right)^3 = Re^{9/4} \]

\[ M / DOF \propto L^{3-9/4} \propto Re^{3/4} \]

---

### Modeling vs Simulation

<table>
<thead>
<tr>
<th>All-sim's</th>
<th>Eddy size</th>
<th>Theory/Model</th>
<th>Compute</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_d )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approaches:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Direct Numerical Simulation (DNS)</td>
<td>D (grid size)</td>
<td>All eddies larger than grid size are computed</td>
<td></td>
</tr>
<tr>
<td>Large Eddy Simulation (LES)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Very Large Eddy Simulation (VLES)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All CR's</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reynolds Averaged Navier-Stokes (RANS)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Principle of Least-Computing!**
Turbulence Models

Algebraic

Differential multi-scalar

Differential tensor

Eddy-Viscosity (Boussinesq, 1870)

\[ \langle u'_i u'_j \rangle \sim \nu_{Eddy} (\nabla_i U_j + \nabla_j U_i) \]

Achille’s heel: No scale separation!
Algebraic Models

Smagorinski

\[ v_{\text{eff}} = v_0 + v_{\text{turbo}}(S) \]

\[ v_{\text{turbo}}(S) = C_s \Delta^2 |S| \]

\[ \frac{v_{\text{turbo}}}{v_0} = C_s \frac{\Delta^2}{\lambda^2} \times (\tau |S|) \gg 1 \]

Algebraic Models

Plus: Simple!

Minus: Too much dissipation at solid walls
Local, No memory
K-epsilon

\[ \tau_{\text{turbo}} = k / \varepsilon \equiv k / (dk / dt) \]

\[ Dk / Dt = \text{div}(v \nabla k) + P_k - R_k \]

\[ D\varepsilon / Dt = \text{div}(v \nabla \varepsilon) + P_\varepsilon - R_\varepsilon \]

K-epsilon

Plus: General

Minus: isotropic, Unstable boundary conditions
Reynolds-Averaged

Tensor: dyn eqs for
Reynolds Stress: \(<u'_i u'_j>\):

Involves Triplets: \(<u'_i u'_j u'_k>\)

Complex and hard on boundaries
Flow Past a Compressor Trailing Edge; Pressure Coefficient $C_p$

Recent from EXA

Complex mesh in matter of
Noise reduction EXA for NASA

Summary

1. Turbulence obeys scaling laws, spatial dimensions very important

2. 3d turbulence supports near-singular structures, 2d supports metastable quasi-particle structures

3. Full scale resolution in 3d is limited to Re=10^4

4. Turbulence modeling required for most practical flows (cars, airplanes, environment ...)

http://www.nas.nasa.gov/SC13/demos/demo2.html
Assignements

1. Derive the expression of the Kolmogorov length in $d=3$ and $d=2$.

2. Show that the non-linear term in $2d$ is zero if the vorticity is a local function of the streamfunction.

3. Write a computer program for 2d flows in Stream-vorticity formulation (periodic boundaries)

For the brave: use spectral methods