Problem Set #3 | Instructor: David Laibson
Due: 25 September 2014 | Economics 2010c

Problem 1 (A simple consumption problem). Consider the following sequence problem: Find $v(x)$ such that

$$v(x_0) = \sup_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to $c_t \in [0, x_t]$, $x_{t+1} = R(x_t - c_t)$ with x_0 given. The associated **Bellman Equation** is given by:

$$v(x) = \sup_{c\in[0,x]} \{\ln(c) + \beta v(R(x-c))\} \quad \forall x.$$

The associated **Functional Equation** is given by:

$$(Bw)(x) = \sup_{c\in[0,x]} \{\ln(c) + \beta w(R(x-c))\} \quad \forall x.$$

We also be interested in the **Finite Horizon Sequence Problem (SP):** Find $v(x)$ such that

$$v_0(x_0) = \sup_{\{c_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \ln(c_t)$$

subject to $c_t \in [0, x_t]$, $x_{t+1} = R(x_t - c_t)$ with x_0 given. We also have the associated finite horizon **Bellman Equation**:

$$v_t(x) = \sup_{c\in[0,x]} \{\ln(c) + \beta v_{t+1}(R(x-c))\} \quad \forall x$$

Note that this finite horizon set-up is the same as the infinite-horizon set-up, except that the value functions are subscripted: $\{v_0(\cdot), v_1(\cdot), v_2(\cdot), ..., v_T(\cdot)\}$.

Let's adopt the following notation:

- $v_T(x) = \ln x$
- Let $v_{T-1}(x) \equiv (Bv_T)(x)$
- More generally, let $v_{T-t-1}(x) \equiv (Bv_{T-t})(x)$

- This notation emphasizes connection between
  - iterating the functional operator B
  - backwards induction
- These two procedures are identical if the initial value function is the value function that applies in the last period of the finite-horizon game (i.e., the initial value function is $\ln x$ for the current application).

a. Using backward induction (starting with $v_T(x) = \ln x$) show that:

$$c_{T-1} = \frac{x}{1+\beta} \equiv \lambda_{T-1} x$$

$$c_{T-2} = \frac{x}{1+\beta+\beta^2} \equiv \lambda_{T-2} x$$

b. Using an induction argument, show that:

$$c_{T-t} = \lambda_{T-t} x$$

where

$$\lambda_{T-t} \equiv \frac{1-\beta}{1-\beta^{t+1}}. \qquad \text{(MPC)}$$

c. Derive the limiting consumption rule

$$c = (1-\beta)x = \lim_{t\to\infty} \lambda_{T-t} x$$

and the limiting value function

$$\lim_{t\to\infty} v_{T-t}(x) \equiv \frac{\ln((1-\beta)x)}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(R\beta).$$

d. Check that our limit function is a fixed point of the original functional equation, check that

$$v(x) = (Bv)(x)$$

I.e., check that

$$v(x) = \sup_{c\in[0,x]} \{\ln(c) + \beta v(R(x-c))\}.$$

e. Plot a few iterations of the policy function and the value function. In other words, backwards induct the policy and value functions from the "seed" functions $c_T(x) = x$ and $v_T(x) = \ln(x)$. Why is the policy function falling with each iteration? How is the value function changing with each iteration? Provide intuition.

Problem 2: True/false/uncertain. Explain your answer.

1. A temporary tax cut would do more to increase the consumption of 25-year-olds than 55-year-olds.
2. If a worker expects a 15% wage increase to occur in 2014, then consumption should rise in anticipation in 2013.
3. The marginal propensity to consume out of a tax rebate should be close to one for housholds that face a binding liquidity constraint.

Problem 3: Three period hyperbolic discounting model.

Consider a person who lives for three periods, $t = 1, 2, 3$. To simplify exposition, I'll refer to three "selves" of this individual. These selves are indexed by their respective periods of control over the individual's consumption decision. At $t = 1$, "self one" chooses c_1. At $t = 2$, "self two" chooses c_2. At $t = 3$, "self three" chooses c_3. At time $t = 1$ the agent has the following utility function:

$$\ln c_1 + \beta\delta \ln c_2 + \beta\delta^2 \ln c_3.$$

These preferences would be "standard" if we set $\beta = 1$, in which case the discount structure would be "exponential." However, we'll assume $0 < \beta < 1$, to approximate the hyperbolic discount structure observed by experimental psychologists. Setting $\beta < 1$ captures the idea that, from the perspective of the current self, the discount between today and tomorrow is sharper then the discount between some future period and the day after that future period. At time $t = 2$ the agent has the following utility function:

$$\ln c_2 + \beta\delta \ln c_3.$$

Again, this utilityfunction reflects the sharp discount between the new current period (i.e., period two) and the new tomorrow (i.e., period three). Finally, at time $t = 3$ the agent has the following utility function:

$$\ln c_3.$$

a. Prove that if $\beta = 1$, then the utility function at $t = 2$ is a linear transformation of the last two terms of the utility function at $t = 1$. Prove that if $\beta < 1$, then the utility function at $t = 2$ is not a linear transformation of the last two terms of the utility function at $t = 1$. Prove that if $\beta < 1$, then the utility function at $t = 2$ is not a monotonic transformation of the last two terms of the utility function at $t = 1$. I.e., show that selves one and two do not have the same rankings over points in (c_2, c_3) space.

b. Suppose self one could choose c_1, c_2, c_3, subject to the constraint, $c_1 + c_2 + c_3 \leq A_1$, where A_1 is the starting asset stock of the agent, and the gross interest rate is implicitly assumed to be equal to one. To simplify notation, assume $\delta = 1$, for this question and all remaining questions in the problem set. Show that under this "precommitment" scenario, self one would choose consumption levels such that

$$u'(c_1) = \beta u'(c_2) = \beta u'(c_3),$$

implying that,

$$\beta c_1 = c_2 = c_3.$$

c. Now suppose that self two is given the chance to revise self one's consumption program for periods two and three. Intuitively, explain why self two has an incentive to implement such a revision (iff $\beta \neq 1$). Assume that self two "inherits" an asset stock of $A_2 = A_1 - c_1$ and is asked to pick c_2 and c_3. Show that self two will choose consumption levels such that

$$u'(c_2) = \beta u'(c_3),$$

implying that,

$$\beta c_2 = c_3.$$

d. Now construct the rational backwards induction solution to this problem. Assume $c_3 = A_3 = A_1 - c_1 - c_2$. Then solve self two's problem, to find that

$$u'(c_2) = \beta u'(c_3),$$

which implies that

$$c_2 = \frac{A_2}{1+\beta}$$

and hence,

$$c_3 = \frac{\beta A_2}{1+\beta}.$$

Now consider the problem of self one:

$$\max_{c_1} \ln c_1 + \beta\delta \ln c_2 + \beta\delta^2 \ln c_3$$

subject to the constraints that future selves maximize their own interests. I.e., they pursue their own equilibrium strategies

$$c_2 = \frac{A_1 - c_1}{1+\beta}$$

$$c_3 = \frac{\beta(A_1 - c_1)}{1+\beta}.$$

You should find that self one sets

$$c_1 = \frac{A_1}{1+2\beta}.$$

e. Confirm that the backwards induction solution satisfies the generalized Euler Equation for the hyperbolic economy:

$$u'(c_t) = R\left[\frac{\partial c_{t+1}}{\partial A_{t+1}}\beta\delta + \left(1 - \frac{\partial c_{t+1}}{\partial A_{t+1}}\right)\delta\right] u'(c_{t+1}).$$

Remember to set $\delta = 1$ and $R = 1$. How does this Euler Equation differ from the Euler Equation in a standard exponential economy. Why doesn't the standard perturbation argument work in a hyperbolic economy? (Hint: the equilibrium path is not first-best optimal. So feasible, welfare-enhancing perturbations can exist.)

f. The equilibrium time path of consumption is pareto-inefficient. I.e., the equilibrium path can be perturbed in a way which makes all three selves strictly better off. Find such a perturbation. Hint: perturb c_1 by -Δ, and perturb c_3 by Δ. Note that this perturbation represents an increase in "thriftiness" in the sense that consumption is being postponed to the future.

g. (Optional: primarily for students who know game theory.) Now think about this dynamic choice problem as a game between the three selves. Show that any consumption path that satisfies the budget constraint can be supported by a Nash equilibrium. (Hint: consider threats of 0 consumption.) Explain why the backwards induction consumption path is the unique subgame perfect equilibrium consumption path. Now think about the infinite horizon version of the model (with $\delta < 1$ and a positive net interest rate). Given what you know about the set of subgame perfect equilibria of the infinite horizon prisoner's dilemma (e.g., the Folk Theorem), speculate about the consequences of extending the horizon to infinity in the consumption "game."

Problem 4: A procrastination problem.

- Assume that an agent is a quasi-hyperbolic discounter with $\delta = 1$ and $0 < \beta < 1$.
- Assume that the agent faces a discrete-time infinite horizon problem.
- The agent needs to complete some task. The agent pays a punishment cost of 1 each period *that begins* with the task still uncompleted.
- If a period begins with the task still uncompleted, the agent pays the punishment cost of 1, and then decides whether to complete the task at a cost of c units of (current) effort.
- Once the task is completed, it remains completed forever and the agent pays no more punishment costs.
- Assume that $c > \beta(1 + c)$. Assume too that commitment is not available (except for question 3.)

a. Is the following strategy an equilibrium strategy for a sophisticated agent? "Complete the task in the current period if it hasn't already been completed." Assume that all selves follow this strategy. Is this an equilibrium?

b. Is the following strategy an equilibrium strategy for a sophisticated agent? "Do not complete the task in the current period if it hasn't already been completed." Assume that all selves follow this strategy. Is this an equilibrium?

c. If the agent could commit herself, when would she complete the task? (For all other questions, assume that commitment is not available.)

d. When would a naive agent complete the task? What would her payoff be in this game?

e. Assume that sophisticates follow a stationary mixed strategy equilibrium. In each period, with probability p the sophisticate completes the task. Explain why $0 \leq p \leq 1$ solves the following system of equations, where V is the *continuation* cost for a state in which the task still uncompleted, and W is the cost function of the current self.

$$\begin{aligned} V &= 1 + pc + (1-p)V \\ W &= \min\{1 + c, 1 + \beta V\} \\ c &= \beta V \quad \text{if } 0 \leq p < 1 \\ c &\leq \beta V \quad \text{if } p = 1 \end{aligned}$$

f. Show that

$$V = \frac{1 + pc}{p}$$

and

$$p = \min\left\{\frac{\beta}{c(1-\beta)}, 1\right\}.$$

g. Explain intuitively – using the economics of the problem – why $V \to 1 + c$ as $p \to 1$.

h. Explain intuitively – using the economics of the problem – why $p \to 1$ as $\beta \to 1$.

i. In equilibrium, how long on average will it take the agent to complete this task. Why might that be described as procrastination?