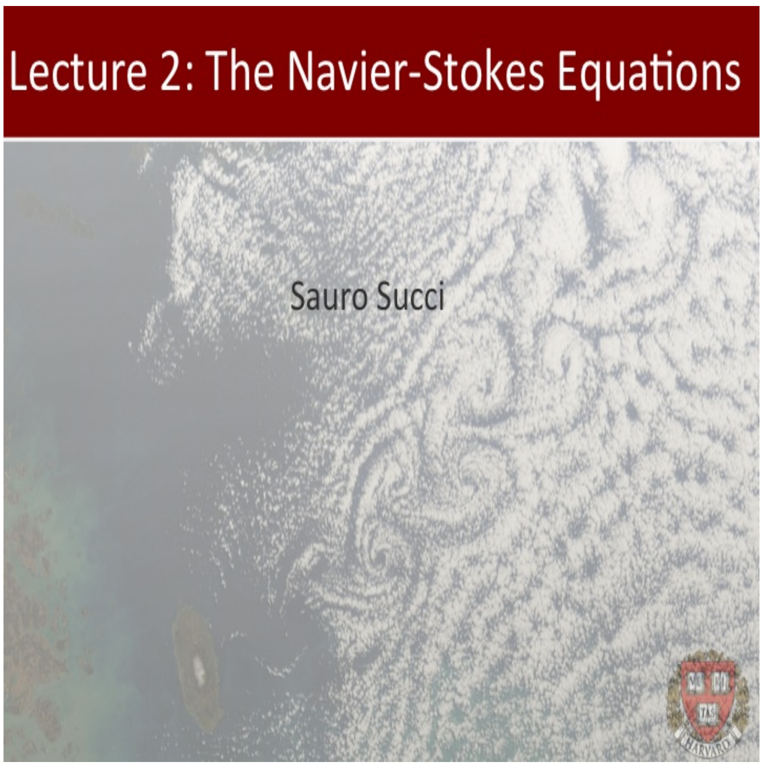
Lecture 2: The Navier-Stokes Equations

September 9, 2015



1 Goal

In this lecture we present the Navier-Stokes equations (NSE) of continuum fluid mechanics.

The traditional approach is to derive teh NSE by applying Newton's law to a finite volume of fluid. This, together with condition of mass conservation, i.e. change of mass per unit time equal mass flux in minus mass flux out, delivers the NSE in *conservative* form, also known as Eulerian form, as it refers to the mass-momentum balance as drawn by an observer at rest. The other approach, known as Lagrangian, corresponds to the picture taken by a co-moving observer (go-with-the flow). Both approaches have merits and pitfalls, but the conservative form is generally more popular, especially for incompressible flows The Navier-Stokes equations are non-linear vector equations, hence they can be written in many different equivalent ways, the simplest one being the cartesian notation. Other common forms are cylindrical (axial-symmetric flows) or spherical (radial flows). In non-cartesian coordinates the differential operators become more cumbersome due to metric terms (inertial forces). For geometries of real-life complexity (cars, airplanes, ...) no global coordinate system can be used, and one resorts to non-coordinate based representations, such as finite-volumes and finite-elements.

2 Eulerian formulation: mass-momentum conservation

The general statement is very simple:

Change per unit time = Flux-in - Flux-out

The relevant quantities are mass-momentum-energy.

The mass in a cube of volume $V = \Delta x \Delta y \Delta z$ is $M = \rho V$ and the flux across the six faces of the cube has the form $J_i = \rho u_i A_i$, where $A_i = \Delta x_j \Delta x_k$, with $i, j, k = x, y, z$.

The mass balance $dM/dt = 0$ applied to the cube of fluid in the limit of zero volume delivers the continutity equation:

$$\partial_t \rho + \partial_i (\rho u_i) = 0 \tag{1}$$

The same argument applied to the momentum in the cube of fluid, $P_i = M u_i$, and taking into account the forces acting on the surfaces, deliver the Navier-Stokes equations.

$$\partial_t \rho u_i + \partial_j (\rho u_i u_j + P \delta_{ij}) = \partial_j \sigma_{ij} + F_i \tag{2}$$

where P is the pressure, σ_{ij} is the stress tensor and F_i the external force per unit volume.

Note that these forces are of two types: i) contact forces, due to the pressure/stress exerted by abutting faces of neighbor cubes, ii) volume forces, due to external fields, say gravity or electric fields for the case of charged fluids. The flux of momentum is a second order tensor, which makes the book-keeping more cumbersome, because there are several contributions to the momentum budget, namely: i) Inertial terms due to the motion of the fluid across the faces of cube, ii) pressure terms due to the component of the contact force normal to the surface it acts upon, iii) stress-terms, due to the tangential component of the force.

To be noted that the latter two require independent inputs, namely an *equation of state* for the pressure as a function of density and temperature, as well as a *constitutive equation* for the stress σ_{ij} as a function of the strain $\nabla_i u_j$.

$$P = P(\rho, T) \tag{3}$$

$$\sigma_{ij} = \mu(\partial_i u_j + \partial_j u_i) + \lambda(\partial_k u_k)\delta_{ij} \tag{4}$$

where μ is the dynamic shear viscosity and λ associates with the bulk viscosity.

Also to be noted that the latter does NOT conserve the total energy of the fluid (kinetic=mechanical+thermal plus potential). These are dissipative terms which emerge from Newtonian mechanics as a result of an approximation

3 Lagrangian formulation

Go-with-the flow, Lagrangian form using the material derivative. The materials derivative runs along the material lines of the flow.

$$\frac{Du_i}{Dt} \equiv \partial_t u_i + u_j \partial_j u_i \tag{5}$$

In a co-moving frame ($u_i = 0$) it reduces to the standard time derivative, which is a major simplification.

$$\frac{D\rho}{Dt} = -\rho \partial_i u_i \tag{6}$$

$$\frac{Du_i}{Dt} = -(\partial_i P)/\rho + (\partial_j \sigma_{ij})/\rho \tag{7}$$

Each blob represents a multitude of molecules: a cc of air contains of the order of 10^{21} molecules (Loschmidt number). This formulation, which is the closest to Newtonian dynamics, is well suited to compressible flows with large deformations (say combustion enegines)

4 General features and family of flows

The NSE are: *multi-dimensional, time-dependent, vectorial, non-linear* and often live in complex geometries (car, airplanes, buildings ...). They are characterized by three main type of forces: Inertial, Pressure, Dissipation. The ratio between these three forces defines the main regimes of fluid flows, characterized by two main dimensionless parameters.

Mach (squared) = Inertia/Pressure;

Reynolds = Inertia/Dissipation

To be reminded that the ratio Ma/Re delivers the Knudsen number, which owes to be very small (less that some percent) if fluid dynamics is to apply at all.

Low/High Mach characterize Incompressible/Compressible flows. Low/High Reynolds characterize Laminar/Turbulent flows Flows at zero Mach are called

Stokes flows, a useful idealization for creeping flows in porous media and many biological applications.

Flows with zero dissipation are called inviscid, a useful idealization for high-Reynolds flows.

5 Compressible/Incompressible

Compressible fluids, typically gases, are characterized by a non-zero divergence of the velocity, i.e the volume occupied by a given amount of mass changes in space and time, so that density is alive. Recall that the divergence of the velocity field describes the change in volume of an element transported by the fluid. Positive(negative) divergence indicates expansion (compression). Zero divergence means that the volume is conserved (shape may change, even dramatically, though). Compressible flows support sound and shock waves.

Sound waves usually carry small perturbations, shock waves carry the large ones, i.e. major density changes across very thin (molecular size) layers.

Incompressiblity is tantamount to infinite sound speed: unphysical but near-true for many fluids which undergo very small changes in density upon huge changes in pressure, parts per thousands or less. Water is a good example and so are most liquids.

$$Ma \to 0$$

Since the information travels at infinite speed, they encode action at distance, which is a tough constraint on numerical methods. Indeed incompressible flows require computational fluid techniques on their own.

6 Viscid/Inviscid

Inviscid fluids have strictly zero viscosity,

$$\nu = 0 \tag{8}$$

hence they do not dissipate (also called perfect fluids).

The NSE with zero viscosity are called Euler equations.

This is a (very useful) idealization, which describes accurately flows away from boundaries. Near boundaries, however, the approximation breaks down, because strong gradients couple even to vanishingly small viscosity and generate dissipation. Dissipation concentrates in thin regions called "boundary layers", often below mm for macroscopic flows in the scale of meters. However, neglecting these layers would be a deadly mistake because all energy is dissipated there. Thus the limit

$$\nu \to 0$$

must be kept very distinct from the strictly inviscid condition $\nu = 0$.

7 Laminar/Turbulent

The Reynolds number measures Inertia/Dissipation. It is typically a large number, easily in the order of millions for ordinary macroscopic bodies. Example: $U = 30$ m/s, $L = 1$ m, $\nu \sim 10^{-5}$ m2/s, gives $Re \sim 30 \times 1 \times 10^5 = 310^6$. Why? It is the ratio between the macroscopic size L and the molecular mean free path, as best appreciated via the Von Karman relation

$$Re = \frac{Ma}{Kn}$$

When the flow is very slow (creeping flows) the NSE are called Stokes equations:

$$Re \to 0$$

Very relevant to biological flows, say cells in blood, bacteria in water... Zero-Reynolds does not necessarily imply slow flows because the non-linearity can be suppressed (depleted) by mere geometrical symmetries, like in Coeutte or Poiseuille flows.

The opposite limit to *fully developed turbulence*, i.e.

$$Re \to \infty$$

.

Turbulent flows are ubiquituous and represent one of the most challenging problems of classical physics and engineering

8 Steady/Unsteady

Unsteady flows show inherent time-dependence (local oscillations), typically driven by the presence of obstacles or external forces. Spontaneous transitions occurr once the system is approaching an instability. Vortex shedding is the precursor of turbulence. Unsteadiness is measured by the Strouhal number, basically the period of the spontaneous oscillations versus the transit time of the flow across the obstacles.

$$St \sim \frac{\partial_t u}{u \nabla u}$$

Typical Strouhal numbers are usually well below 1, indicating that inertia usually prevails on autonomous oscillations.

9 Newtonian/Non Newtonian

This refers to the relation between appled stress σ_{ij} and resulting strain $D_{ij} \equiv \partial_i u_j$. Linear proportionality characterizes Newtonian fluids.

10 Ideal/Nonideal

When the equation of state is not proprtional to the density (potential energy contribution)

Ideal:

$$P = nRT = \rho v_T^2$$

Non-Ideal

Typical van-der Waals, short-range repulsuon, long range attraction:

$$(P - a/V^2)(V - b) = nRT$$

11 Some simple flows

Incompressible flows: Couette flow (constant shear) and Poiseuille flow (linear shear).

12 Boundary Conditions

Boundary conditions select the solutions compatible with the environmental constraints. They depend both on the environment and the inherent nature/regime of the flow. In actual practice, they are the most critical factor in the development of robust and efficient CFD methods.

Among others:

- Solid walls: no-slip velocity
- Inlets: imposed density (pressure) and velocity
- Outlets: imposed density(pressure) and zero normal gradient
- Symmetry Planes: zero normal gradient

13 Summary

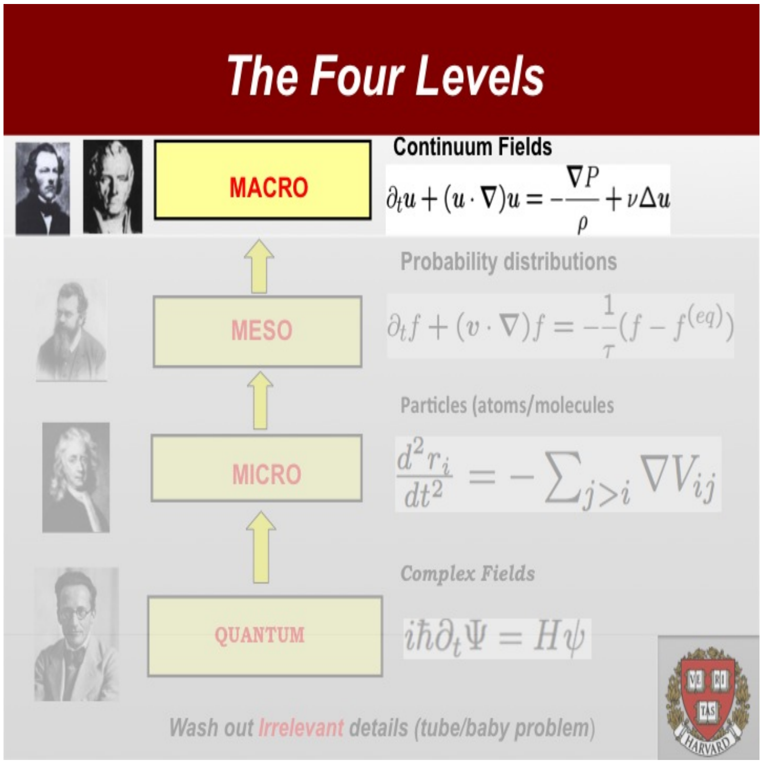
Summarizing, the Navier-Stokes equations of continuum fluid mechanics are "simply" Newton's law $ma = F$ as applied to a small volume of fluid. Despite their elementary physical meaning, they prove exceedingly difficult to solve, as they assemble three nigthmares of computational physics: strong non-linearity, complex geometry, fully three-dimensional, time-dependent configurations.

14 Exercises

1. Prove the equivalence between Lagrangian and Eulerian formulations

2. Derive the equations for incompressible flows from the general NSE's. Hint: Use the identities $\nabla \cdot (\rho \vec{u}) = \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u}$ and $\nabla \cdot (\rho \vec{u} \vec{u}) = \vec{u} \cdot \nabla (\rho \vec{u}) + \rho \vec{u} \cdot \nabla \vec{u}$

3. Prove the von Karman relation.


The Four Levels
MACRO
Continuum Fields
$\partial_t u + (u \cdot \nabla) u = -\frac{\nabla P}{\rho} + \nu \Delta u$
MESO
Probability distributions
$\partial_t f + (v \cdot \nabla) f = -\frac{1}{\tau}(f - f^{(eq)})$
MICRO
Particles (atoms/molecules
$\frac{d^2 r_i}{dt^2} = -\sum_{j>i} \nabla V_{ij}$
QUANTUM
Complex Fields
$i\hbar\partial_t \Psi = H\psi$
Wash out Irrelevant details (tube/baby problem)
HARVARD

Eulerian versus Lagrangian

Eulerian: Observer at rest watches the fluid go by;
Conservative form

Lagrangian: Co-moving observer (go-with-the-flow);
Non-conservative form

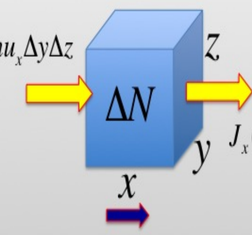
Eulerian: Conservation Laws

Mass Balance

Mass time change= Flux_in-Flux_out

$J_x(west) = nu_x \Delta y \Delta z$

Molecules/time across surface



$$\Delta N = n\Delta x \Delta y \Delta z$$

$J_x(east) = nu_x \Delta y \Delta z$

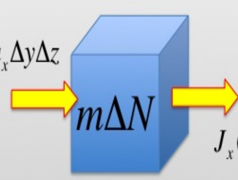
$$\delta(\Delta N)_x = \{J_x(x - \Delta x/2) - J_x(x + \Delta x/2)\}\delta t$$
$$\delta(\Delta N)_y = \{J_y(y - \Delta y/2) - J_y(y + \Delta y/2)\}\delta t$$
$$\delta(\Delta N)_z = \{J_z(z - \Delta z/2) - J_z(z + \Delta z/2)\}\delta t$$

Continuity equation

In the limit of zero V and dt:

$J_x(west) = \rho u_x \Delta y \Delta z$

Mass/time across surface

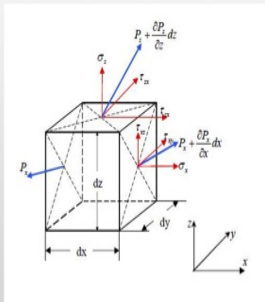


$$\rho = nm$$

$J_x(east) = \rho u_x \Delta y \Delta z$

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0}$$

NSE from Newton:



$$m\vec{a} = \vec{F}$$

Volume + contact forces

$$\vec{f} = \rho\vec{g} + div\,\overleftrightarrow{\Pi}$$

$$\oint(\rho\vec{a})\,dV = \oint\vec{f}\,dV$$

$$n = \rho/m = N/V$$

Momentum Flux: Inertia

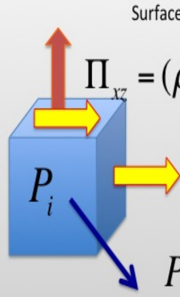
X-Momentum/time across
Surface with normal along Z

$$A_x = \Delta y \Delta z$$

$$A_y = \Delta x \Delta z$$

$$A_z = \Delta x \Delta y$$

$$u_i = (u,v,w) = (u_x,u_y,u_z)$$



$$\Pi_{xz} = (\rho u_x A_z)u_z$$

$$\Pi_{xx} = (\rho u_x A_x)u_x$$

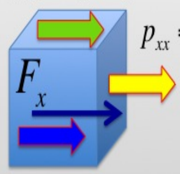
$$P_i = \rho u_i \Delta x \Delta y \Delta z$$

$$\delta P_i = \{\Pi_{ix}(west) - \Pi_{ix}(east) + ...y... + ...z + F_i\}\delta t$$

Momflux: Forces

Contact Forces

$p_{xz} = F_x / A_z$



$p_{xx} = F_x / A_x$

p_{xy}

$A_x = \Delta y \Delta z$

$A_y = \Delta x \Delta z$

$A_z = \Delta x \Delta y$

$$F_x = [p_{xx} A_x]_{x<}^{x>} + [p_{xy} A_y]_{y<}^{y>} + [p_{xz} A_z]_{z<}^{z>}$$

NSE Cartesian

$$\partial_t(\rho u) + \partial_x(\rho uu) + \partial_y(\rho uv) = \partial_x(\sigma_{xx} - p) + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} + f_x$$

$$\partial_t(\rho v) + \partial_x(\rho vu) + \partial_y(\rho vv) = \partial_x \sigma_{yx} + \partial_y(\sigma_{yy} - p) + \partial_z \sigma_{zx} + f_y$$

$$\partial_t(\rho w) + \partial_x(\rho wu) + \partial_y(\rho wv) = \partial_x \sigma_{zx} + \partial_y \sigma_{yx} + \partial_z(\sigma_{zz} - p) + f_y$$

$p_{xx} = p$ Pressure

$p_{ij} = \sigma_{ij}$ Stress

$m\vec{a} = \vec{F}$

Conservative-NSE

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0 \qquad D_t(\rho V) = 0$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u}\vec{u} + \overleftrightarrow{P}) = 0 \qquad D_t(\rho \vec{u} V) = \vec{F}$$

$$\overleftrightarrow{P} = p\overleftrightarrow{I} - \vec{\sigma}$$

$$m\vec{a} = \vec{F}^{(I)} + \vec{F}^{(P)} + \vec{F}^{(D)}$$

(pseudo-Newtonian form)

More than Newton

Eos+Constitutive: (assumptions)

$$p = f(\rho, T)$$

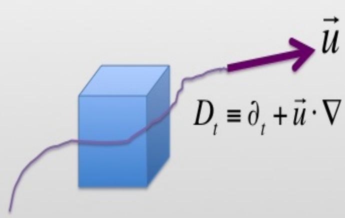
$$\vec{\sigma} = \lambda(\nabla \cdot \vec{u})\overleftrightarrow{I} + \mu(\vec{\nabla}\vec{u})$$

Dissipative do not conserve energy

$$h = \frac{\rho u^2}{2} + p + n\Phi \qquad (F^{ext} = -\nabla\Phi)$$

Lagrangian Formulation

Navier-Stokes from Newton

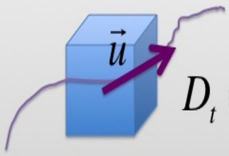


$$D_t \equiv \partial_t + \vec{u} \cdot \nabla$$

$$M = \rho V \qquad D_t(\rho V) = 0$$

$$M\vec{a} = \vec{F} \qquad D_t(\rho \vec{u} V) = \vec{F}$$

Navier-Stokes from Newton



$$D_t \equiv \partial_t + \vec{u} \cdot \nabla$$

$$M = \rho V \qquad D_t(\rho V) = 0$$

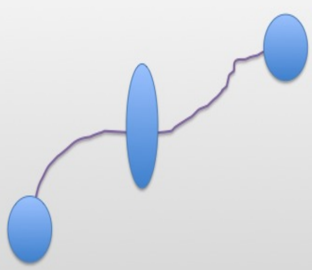
$$M\vec{a} = \vec{F} \qquad D_t(\rho \vec{u} V) = \vec{F}$$

Lagrangian-NSE

$$D_t \rho = -\rho(\nabla \cdot \vec{u})$$

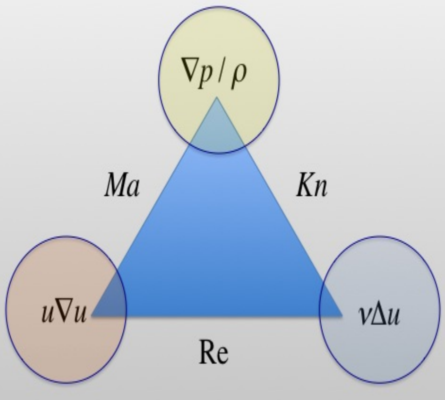
$$D_t \vec{u} = -\frac{\nabla p}{\rho} + \frac{\nabla \cdot \overleftrightarrow{\sigma}}{\rho}$$

$$D_t \equiv \partial_t + \vec{u} \cdot \nabla$$



Three actors on stage

PID = Pressure-Inertia-Dissipation



Flow regimes

Fluid vs rarefied gases
Incompressible/Compressible
Inviscid/Viscous
Laminar/Turbulent

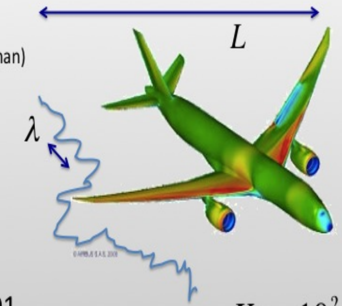
Steady/Unsteady
Ideal/Non-ideal
Newton/Non-Newton

To fluid or not fluid?

Knudsen number: Dissipation/sqrt(Pressure*Inertia)

$$Kn = \frac{Ma}{Re}$$ (Macro: van Karman)

$$\boxed{Kn = \frac{\lambda}{L}}$$ (micro)



$$Kn \approx 10^{?}$$

Standard fluid: Kn<0.01
Slip: 0.01<Kn<0.1
Transition: 0.1<Kn<1
Wild bunch (rarefied gas): Kn>1

Compressible/Incompressible

$$Mach \equiv \frac{u}{c_s}$$

$Incompressible \quad Ma \to 0$

$Compressible:$

$Sub/supersonic \quad Ma < 1, \quad > 1$

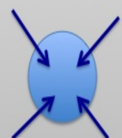
Compressible/Incompressible

$$Ma^2 = \frac{\rho u \nabla u}{\nabla p} \approx \frac{u^2}{c_s^2}$$

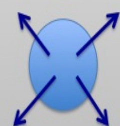
$$D_t\rho \equiv \partial_t\rho + u\nabla\rho = -\rho\, div\, u$$

$$div\,\vec{u} \equiv \frac{\delta V / V}{\delta t}$$

$div\, u < 0$

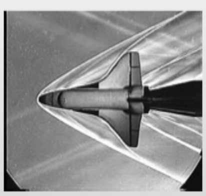


$div\, u > 0$



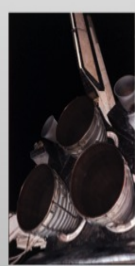
Flow Regimes

Compressible



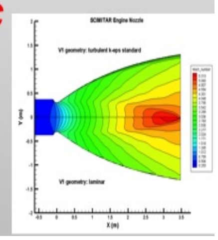
$$\partial_t\rho + \nabla\cdot(\rho\vec{u}) = 0$$

$$div\,\vec{u} \neq 0$$



Sub/Supersonic

$$Mach \approx \frac{u}{c_s} < 1$$

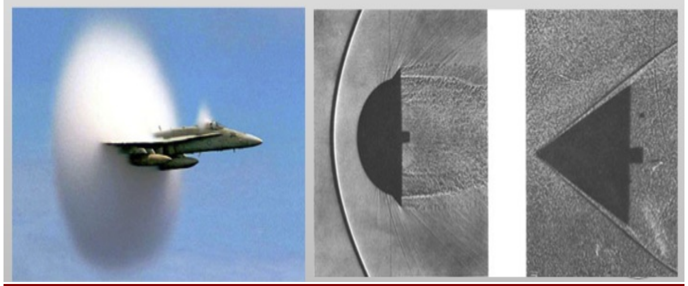


Compressible: sound waves

$$\left|\frac{\delta\rho}{\rho}\right| \approx Ma^2 \left|\frac{\delta u}{u}\right|$$

$$Subsonic: \quad Ma < 1: \quad pressure > inertia$$

$$Supersonic: \quad Ma > 1: \quad pressure < inertia$$



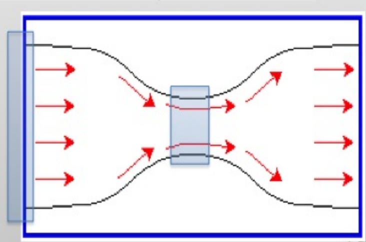
Incompressible-NS

$$\rho = const.$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla P}{\rho} + \nu \Delta \mathbf{u}$$

$$\nabla \cdot \vec{u} = 0$$

$$u / c_s \to 0$$



$$u_x A = const.$$

Incompressible-NS

Non-local pressure (sound speed 2 infinity)

$$\nabla \cdot (\vec{u} \cdot \nabla \vec{u}) = -\Delta p$$

$$\Delta p = -\nabla \cdot \vec{A} \equiv q$$

No EoS needed: kinematic constraint

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} = -\frac{\boldsymbol{\nabla} P}{\rho} + \nu \Delta \boldsymbol{u}$$

Inviscid/Viscous

Kinematic viscosity (m^2/s): $\nu \approx \lambda \sqrt{kT / m}$

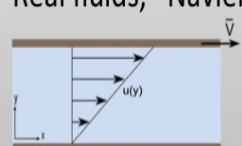
$\nu = V_{th}^2 / \tau$

Inviscid: $\nu = 0$

Perfect fluids, Euler equation (superfluids)

Viscous: $\nu > 0$

Real fluids, Navier-Stokes equations



Laminar/Turbulent

$$\mathrm{Re} = \frac{u\nabla u}{\nu \Delta u} = \frac{UL}{\nu}$$

Laminar: $\mathrm{Re} \to 0$

High visco, small, slow (Biology)

Turbulent: $\mathrm{Re} \to \infty$

Low visco, large, fast (Aeronautics, Environment)

Turbojet

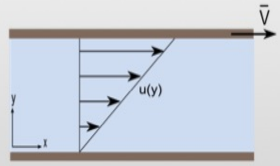
Non-linear depletion

Nominal Re>0, but effective Re=0 because nonlinearity is depleted by geometrical symmetry

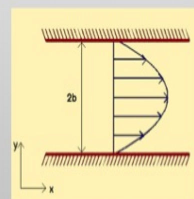
$$(\mu = \rho\nu > 0)$$

$$\vec{A} \equiv \vec{u} \cdot \nabla \vec{u} = 0$$

Couette flow



Poiseuille flow

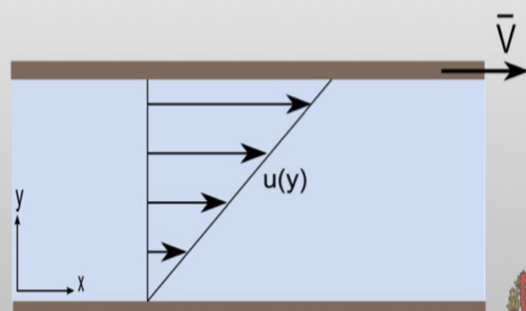


Local pancakes: 2D

Couette Flow

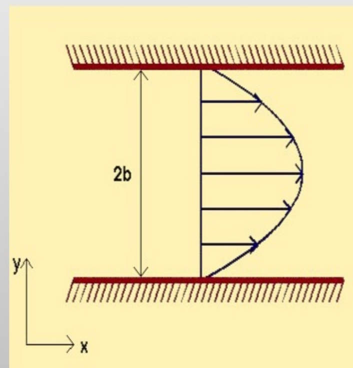
$$u(x,H) = V \qquad u(x,0) = 0$$

$$\nabla p = 0$$



Poiseuille flow

$$\mu\partial_y^2 u_x = -\nabla_x p \qquad u_y = 0$$

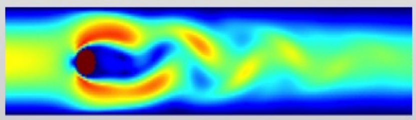


Steady/Unsteady

$$St \approx \frac{\partial_t u}{u\nabla u} \to 0$$

Vortex shedding

$$St \approx \frac{\partial_t u}{u \nabla u} = \frac{fD}{U}$$



Equation of State

Ideal:

Pressure linear with density
Simple molecules

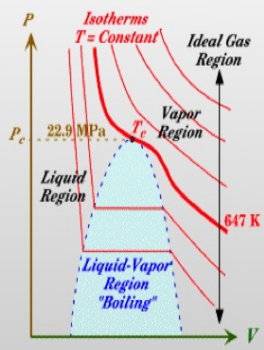
$$pV = n_m RT$$

Non-ideal:

Pressure non linear with density:
(complex fluids, internal structure)

$$pV / n_m RT = 1 + Z(n)$$

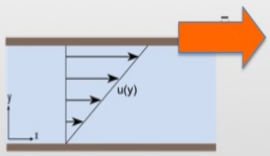
p



V

Rheology

Stress: $\sigma = F / A$



$$\sigma_{ij} = F_i / A_j$$

Strain: $S_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

Rheology

Newtonian:

Strain linear with Stress
Simple molecules

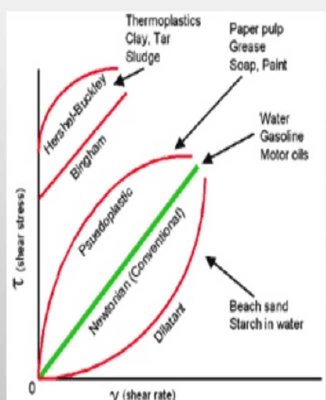
$$\sigma = \mu S$$

Non-Newtonian:

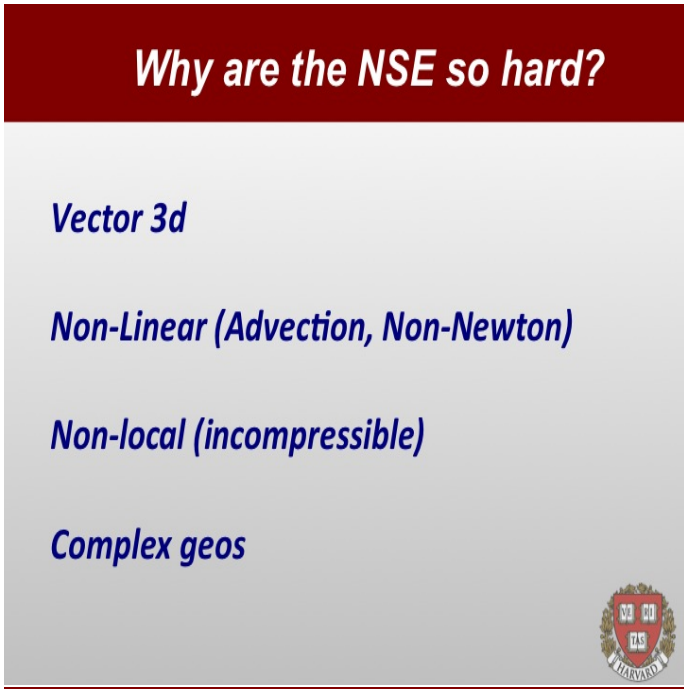
Strain vs Stress: nonlinear, nonlocal in space and time
(complex fluids, internal structure)

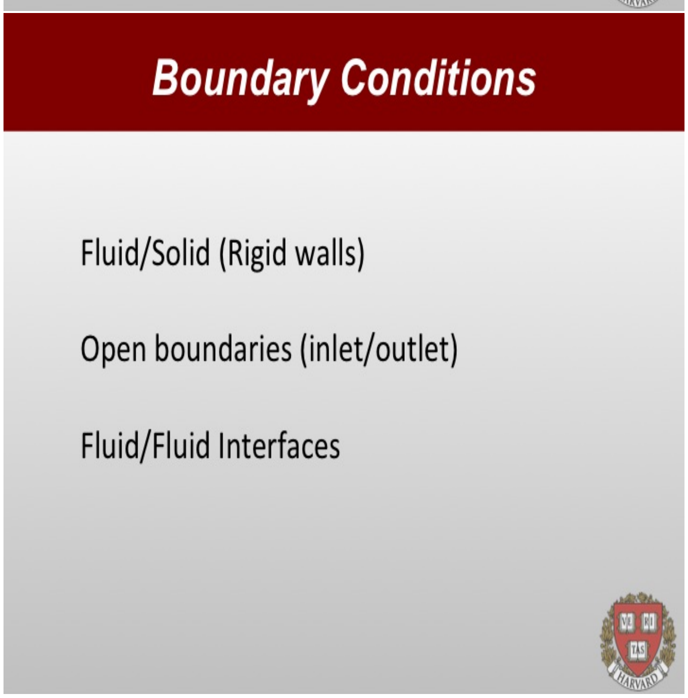
$$\sigma = \mu(S) S$$

σ



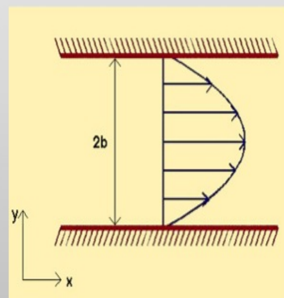
S


Why are the NSE so hard?
Vector 3d
Non-Linear (Advection, Non-Newton)
Non-local (incompressible)
Complex geos


Boundary Conditions
Fluid/Solid (Rigid walls)
Open boundaries (inlet/outlet)
Fluid/Fluid Interfaces

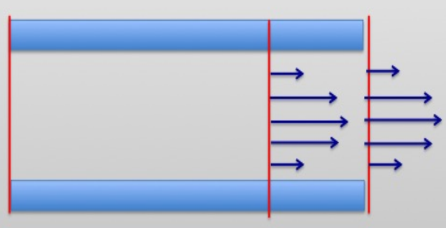
Solid walls: No-slip

$$u = U_{wall} = 0$$
$$v = V_{wall} = 0$$



Open flows

$$u(x=0) = u_{inl} \qquad \frac{\partial u}{\partial n}\Big|_{x=L} = 0$$
$$v(x=0) = V_{inl}$$
$$p(x=0) = P_{out} \qquad p(x=L) = P_{out}$$



Assignements

Derive the NS eq.s for incompressible flows

Derive the analytical expression of Poiseuille flow in 2d
(show that nonlinear terms are identically zero)

Write a 2d computer program to solve the Poiseuille equations
(see matlab codlets)

End of Lecture 2