Ma1a Summary Sheet

1. The Principle of Induction: Let $\{P(n)\}$ be a sequence of statements running over the natural numbers. Suppose that P(1) is true and suppose that if P(n) is true, it follows that P(n+1) is true. Then P(n) is true for all natural numbers n.

2. **Well ordering principle:** Every nonempty set of natural numbers has a smallest element.

3. Ordering of real numbers: Given two real numbers x and y, either $x \geq y$ or $y \geq x$.

4. Any nonempty set of real numbers A which has a real upper bound, has a least upper bound in the reals.

5. **Definition of the limit:** A sequence $\{a_n\}$ converges to limit L if for every real number $\epsilon > 0$ there is a natural number N so that $|a_n - L| < \epsilon$ whenever $n > N$.

6. Fundamental Theorem of Analysis: Every bounded monotonic sequence of real numbers converges.

7. Convergence: A monotonic sequence converges if and only if it is bounded.

8. If L is the least upper bound of an increasing sequence of real numbers bounded above, the sequence converges to L.

9. **Cauchy Sequence:** A sequence is Cauchy provided that for every $\epsilon > 0$, there is a natural number N so that when $n, m \geq N$, we have $|a_n - a_m| \leq \epsilon$. A sequence converges if and only if it is Cauchy.

10. Subsequence test for convergence: If a sequence converges to some value L, then all of its subsequences also converge to L.

11. **Archimedean Principle:** For all $x \in \mathbb{R}$, there exists a $N \in \mathbb{N}$ such that $N \geq x$. That is, there is no maximum element in the reals.

12. **Bolzano-Weierstrass Theorem:** All bounded sequences of real numbers have a convergent subsequence.

13. **Squeeze theorem:** Given three sequences of real number a_n, b_n, c_n, if a_n and b_n both converge to the same limit L, and if we know that $a_n \leq c_n \leq b_n$, then c_n converges to limit L as well.

14. **Infinite squeeze theorem** If a_n is a sequence of positive real numbers going to infinity, and $b_n \geq a_n$, then the sequence b_n converges to infinity.

15. Tails of convergent series: The series $\sum_{n=1}^{\infty} a_n$ converges if and only if its tail $\sum_{n=M}^{\infty} a_n$ converges.

16. If a_n, b_n are two sequences of real numbers, and if $0 \leq a_n \leq b_n$, if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges as well.

17. **Absolute convergence:** A series is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

18. **Alternating series test:** If $\{a_n\}$ is a monotonically decreasing sequence to zero, then $\sum_{n\to\infty} (-1)^n a_n$ converges. Note that it is not sufficient for the sequence to be decreasing. It must decrease to zero.

19. **Ratio test:** Suppose $a_n \neq 0$ for any n sufficiently large. Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $L > 1$, the series diverges.

20. **nth root test:** Suppose $\lim_{n\to\infty} |a_n|^{1/n} = L$. If $L < 1$, the series $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $L > 1$, the series diverges.

21. nth term test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

22. Radius of convergence: Consider the power series $S(x) = \sum_{j=0}^{\infty} a_j x^j$. There is a unique $R \in [0, \infty]$ such that $S(x)$ converges absolutely when $|x| < R$ and diverges when $|x| > R$. R can be infinity.

23. Sum of series: If $S_a = \sum_{n=0}^{\infty} a_n$ and $S_b = \sum_{n=0}^{\infty} b_n$ are both absolutely convergent, then $S_a + S_b = \sum_{n=0}^{\infty} a_n + b_n$, and $S_a S_b = \sum_{m=0}^{\infty} c_m$, with $c_m = \sum_{i+j=m} a_i b_j$.

24. Limit Laws: If the limit of a_n and b_n exists and is L_1 and L_2 respectively, then $\lim_{n\to\infty} a_n + b_n = L_1 + L_2$, and $\lim_{n\to\infty} a_n b_n = L_1 L_2$. If $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L_1}{L_2}$.

25. Function: A function f from the reals to the reals is a set G of ordered pairs (x, y) so that for any real number x, there is at most one y with $(x, y) \in G$. The set x for which there is a y for which $(x, y) \in G$ is called the domain of the function. If x is in the domain, the real number y for which $(x, y) \in G$ is called $f(x)$.

26. **Limit of function:** We say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ there is $\delta > 0$ so that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

27. Squeeze theorem for functions: Let f,g,h be functions defined on the reals without the point a. Suppose that $f(x) \leq h(x) \leq g(x)$ and suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$. Then $\lim_{x \to a} h(x) = L$.

28. **Continuity:** A function on the reals is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$.

29. Negation of continuity: A function on the real is not continuous if there is some value of $\epsilon > 0$ for which we cannot find a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

30. **Extreme value theorem:** Let $f(x)$ be a function which is continuous on the interval $[a, b]$. Then $f(x)$ attains its maximum on this interval. If $M = l.u.b.\{f(x) : x \in [a, b]\}$ then M exists and there is a point $c \in [a, b]$ so that $f(c) = M$.

31. Intermediate value theorem: Let f be a function on the interval $[a, b]$. Suppose that $f(a) < L < f(b)$. Then there is some $c \in [a, b]$ so that $f(c) = L$.

32. **Little oh:** A function $f(h)$ is $o(h)$ if as $h \to 0$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$. A function $f(h)$ is $o(g(h))$ if $\lim_{h \to 0} \frac{f(h)}{g(h)} = 0$ if $g(h)$ is a continuous increasing function of h with $g(0) = 0$.

33. **Big oh:** A function f is $O(h)$ as $h \to 0$ if there exsts $C, \delta > 0$ so that for $|h| < \delta$, then $|f(h)| \leq C|h|$. A function $f(h)$ is $O(g(h))$ if there exists $C, \delta > 0$ so that for $|h| < \delta$, we have $|f(h)| \leq Cg(|h|)$, if $g(h)$ is a continuous increasing function of h with $g(0) = 0$.

34. **Differentiability:** A function f is differentiable at x if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$ exists.

35. Function class: A function $f : \mathbb{R} \to \mathbb{R}$ is a C^k function on a specified interval if it has k continuous derivatives on that interval. That is, it is k times continuously differentiable. A C^0 function is a continuous function.

36. First order differential approximation: $f(x + h) = f(x) + hf'(x) + o(h)$. $f(x + h) = f(x) + O(h)$.

37. Mean value theorem: Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and which is differentiable at every point of the interior (a, b). Then there is a point $c \in (a, b)$ so that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

38. If a function f is continuous on the interval $[a, b]$ and differentiable at every point of the interior (a, b). Suppose that $f'(x) > 0$ for every $x \in (a, b)$, then $f(x)$ is strictly increasing on $[a, b]$.

39. Inverse Rule: Suppose $f(g(x)) = x$ and g is differentiable at x with nonzero derivative and f is differentiable at $g(x)$ then $f'(g(x)) = \frac{1}{g'(x)}$.

40. First Derivative Test: Let a function be continuous on the closed interval $[a, b]$ and differentiable on the interior (a, b). Let $c \in (a, b), f'(c) = 0$. Suppose there is some $\delta > 0$ such that $\forall x \in (c - \delta, c)$, we have that $f'(x) > 0$ and for every $x \in (c, c + \delta), f'(x) < 0$. Then f has a local maximum at c.

41. Second Derivative Test: Let f be a function continuous on the closed interval $[a, b]$, and differentiable on the interior (a, b). Let c be a point $c \in (a, b)$ where $f'(c) = 0$. Suppose the derivative $f'(x)$ is differentiable at c and that $f''(c) < 0$. Then f has a local maximum at c.

42. Taylor's Theorem: Let f be a function continous on a closed interval I having c on its interior. Suppose that $f'(x) \cdots f^{(m-2)}(x)$ are defined and continuous everywhere inside I. Suppose that $f^{(m+1)}$ is defined everywhere on I and that $f^{(m)}$ is defined. Then for h sufficently small such that $[c, c+h] \subset I$, we have: $f(c+h) = f(c) + \sum_{k=1}^{m} \frac{h^k}{k!} f^{(k)}(c) + o(h^m)$

43. Definition of rational powers: Let $x \in \mathbb{R}, \frac{p}{q} \in \mathbb{Q}$. Then $x^{\frac{p}{q}} = l.u.b.\{y : y^q < x^p\}$

44. Definition of real powers: Let $x, \alpha \in \mathbb{R}$. Then $x^\alpha = l.u.b.\{x^{\frac{p}{q}} : \frac{p}{q} \in \mathbb{Q}, \frac{p}{q} < \alpha\}$. Note that this requires $x > 1$ so that x^α is increasing.

45. Continuity of powers: Let $k \in \mathbb{R}$. $f(\alpha) = k^\alpha$ is continuous at every real α. Use Cauchy condition to prove limit exists and converges to k^α for a fixed α.

46. Definition of e: $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. Also, $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$.

47. Exponential is faster than polynomial: $\lim_{x \to \infty} \frac{x^k}{e^x} = 0$.

48. Function with zero Taylor series at $x = 0$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

49. $e^{-\frac{1}{x^2}}$ is $o(x^n)$ for all n. So is $e^{\frac{-1}{x}}$. Proceed by changing variables $y = \frac{1}{x}$ and taking the limit as y goes to infinity.

50. Weird infinitely continuous function: Let $f_{[a,b]}(x) = f(x-a)f(b-x)$. Then on the closed interval $[a,b]$, this function is of the class $C^\infty(\mathbb{R})$ with $f_{[a,b]} > 0$ for $x \in (a,b)$ but $f_{[a,b]}(x) = 0$ otherwise.

51. Theorem of Borel: Every power series is the formal Taylor series of some $C^\infty(\mathbb{R})$ function. Let $\sum_{n=0}^\infty a_n x^n$ be some power series. There is a $C^\infty(\mathbb{R})$ function g which has this series as its Taylor series at 0.

52. Newton's method: Let I be an interval and f a function which is twice continuously differentiable on I. Suppose that for every $x \in I$, we have $|f''(x)| < M$ and $|f'(x)| > \frac{1}{K}$. Then if we pick $x_0 \in I$, and define the sequence $\{x_j\}$ by:

$$x_j = x_{j-1} - \frac{f(x_{j-1})}{f'(x_{j-1})}$$

Then if every x_j is in I and $|f(x_0)| < \frac{r}{K^2 M}$, we have the estimate:

$$|f(x_j)| \leq \frac{r^{2^j}}{K^2 M}$$

53. Inverses: Let f be continuous and strictly increasing on $[a,b]$. Then f has an inverse uniquely defined from $[f(a), f(b)]$ to $[a,b]$.

54. Greatest lower bound: Given a set A of real numbers bounded below, the $g.l.b.$ is given by $g.l.b.(A) = -l.u.b.(-A)$.

55. Riemann upper sum: Let P be a partition of an interval $[a,b]$ with a set of points $\{x_0, \ldots, x_n\}$ so that $x_0 = a < x_1 < x_2 \ldots < x_{n-1} < x_n = b$. Then $U_p(f) = \sum_{j=1}^n l.u.b.\{f(x) : x_{j-1} \leq x \leq x_j\}(x_j - x_{j-1})$.

56. Riemann lower sum: $L_p(f) = \sum_{j=1}^n g.l.b.\{f(x) : x_{j-1} \leq x \leq x_j\}(x_j - x_{j-1})$

57. Refinement: A partition Q refines a partition P provided that $P \subset Q$.

58. Lower and upper sums under refinement: Let Q be a partition which refines P. Then for any bounded f defined on $[a,b]$, we have:

$$L_P(f) \leq L_Q(f) \leq U_Q(f) \leq U_P(f)$$

59. Lower sums always smaller or equal to upper sums: Let P and Q be any partitions of $[a,b]$. Then for any bounded f on $[a,b]$, we have $L_P(f) \leq U_Q(f)$.

60. Lower and Upper Integrals: $I_{l,[a,b]} = l.u.b.\{L_P(f)\}$, $I_{U,[a,b]} = g.l.b.\{U_P(f)\}$

61. Riemann integrability: f is Riemann integrable on $[a,b]$ if and only if the lower and upper integrals are equal.

62. Alternative definition of integrability: A function f is integrable on the interval $[a,b]$ if and only if for any $\epsilon > 0$, there is a partition $a = x_0, \ldots x_n$ such that:

$$\sum_{i=1}^{n} \sup_{x \in (x_{i-1}, x_i)} f(x)(x_i - x_{i-1}) - \sum_{i=1}^{n} \inf_{x \in (x_{i-1}, x_i)} f(x)(x_i - x_{i-1}) < \epsilon$$

63. Shifting integral limits: $\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$

64. Scaling integrals: $\int_a^b f(x)dx = \frac{1}{k}\int_{ka}^{kb} f\left(\frac{x}{k}\right)dx$

65. Comparing upper and lower sums: $U_P(c_1 f + c_2 g) \leq c_1 U_P(f) + c_2 U_P(g)$ and $L_P(c_1 f + c_2 g) \geq c_1 L_P(f) + c_2 L_P(g)$. Because the maximum and minimum of f may not coincide with the maximum and minimum of g.

66. Uniformly continuity: A function on the interval $[a,b]$ is uniformly continuous if for every $\epsilon > 0$ there is $\delta > 0$ so that whenever $|x - y| < \delta$, we have that $|f(x) - f(y)| < \epsilon$. Note that this has no fixed x. Hence this applies for any x and any y.

67. A function f on $[a,b]$ which is uniformly continuous is Riemann integrable.

68. If f is continuous on $[a,b]$ and differentiable at every point of (a,b), and if f' is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.

69. **Fundamental Theorem of Calculus:** Let F be a continuous function on the interval $[a,b]$. Suppose F is differentiable everywhere in the interior of the interval with derivative f which is Riemann integrable. Then $\int_a^b f(x)dx = F(b)-F(a)$.

70. Fundamental Theorem of Calculus II: Let f be continuous on $[a,b[$ and let $F(x)=\int_a^x f(y)dy$. Then $F'(x)=f(x)$.

71. Change of variables formula: Let f be integrable on an interval $[a,b]$. Let $g(x)$ be a differentiable function taking the interval $[c,d]$ to the interval $[a,b]$ with $g(c)=a$ and $g(d)=b$. Then $\int_a^b f(x)dx = \int_c^d f(g(x))g'(x)dx$.

72. Improper Integral: If f is bounded and integrable on all intervals of non-negative reals, then $\int_0^\infty f(x)dx = \lim_{y\to\infty}\int_0^y f(x)dx$. If f is bounded and integrable on all intervals $[a,y]$ with $y<b$, then $\int_a^b f(x)dx = \lim_{y\to b}\int_a^y f(x)dx$. These integrals only converge if the limit defining them converges.

73. Integral test for convergence of series: Let f be a decreasing, nonnegative function of positive reals. Then $\sum_{j=1}^\infty f(j)$ converges if $\int_1^\infty f(x)dx$ converges. The series diverges if $\int_1^\infty f(x)dx$ diverges.

74. Midpoint method for numerical integration: Partition the interval $[a,b]$ into n equally spaced subintervals. Take m_j to be the midpoint of the ith interval. Then $J=\sum_{j=1}^n \frac{b-a}{n} f(m_j)$. This has error $|J-\int_a^b f(x)dx| = O(\frac{1}{n^2})$.

75. Trapezoid rule: $J=\sum_{j=1}^n \frac{f(x_{j-1})+f(x_j)}{2}\frac{b-a}{n}$. This has error $O(\frac{1}{n^2})$.

76. Simpson's Rule:

$$J=\sum_{j=1}^n \frac{f(x_{j-1})+4f(m_j)+f(x_j)}{6}\frac{b-a}{n} = \frac{1}{3}J_{Trapezium}+\frac{2}{3}J_{Midpoint}$$

77. Taylor Theorem with Remainder:

$$f(x)=\sum_{j=0}^n \frac{f^{(j)}(c)}{j!}(x-c)^j + \frac{1}{n!}\int_c^x (x-y)^n f^{(n+1)}(y)dy$$

78. Mean value theorem for integrals: Let f and g be continuous functions on the closed interval $[a,b]$. Assume that g does not change sign on $[a,b]$. Then there is $c\in(a,b)$ such that:

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

79. Taylor's Theorem with Mean value theorem: There is some $d\in(c,x)$ so that:

$$R_n(x)=\frac{f^{(n+1)}(d)(x-c)^{n+1}}{(n+1)!}$$
$$f(x)=\sum_{j=0}^n \frac{f^{(j)}(c)}{j!}(x-c)^j + \frac{f^{(n+1)}(d)(x-c)^{n+1}}{(n+1)!}$$

80. Arclength: Let f be a differentiable function on the interval $[a,b]$. Then the arclength of the graph of f is:

$$\int_a^b \sqrt{1+f'(x)^2}dx$$

81. Arcsin:

$$\arcsin a = \int_0^a \frac{dx}{\sqrt{1-x^2}}$$
$$\frac{\pi}{2} \equiv \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

82. Symmetry of sin and cos: $\cos(\frac{\pi}{2}-x)=\sin x, \sin(\frac{\pi}{2}-x)=\cos x$. Replace x with $\frac{\pi}{2}-x$ to obtain the other.

83. Critical point: Let f be a once continuously differentiable function on an interval I, and x be a point in the interior of I. x is a critical point of f if $f'(x)=0$.

84. **Concavity:** A function $f(x)$ is concave if for any a, b, x with $x \in (a,b)$, we have:

$$f(x) \geq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$

A function is also concave if for any a, b and $\lambda \in [0,1]$,

$$f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda)f(b)$$

85. Concavity and second derivative: Let f be twice continuously differentiable. Then f is concave if and only if for every x, we have $f''(x) \leq 0$ and convex if and only if for every x we have $f''(x) \geq 0$.

86. **Means:** Arithmetic geometric mean inequality: $a^{\frac{1}{2}}b^{\frac{1}{2}} \leq \frac{1}{2}(a+b)$. Generalized inequality: $a^{\alpha}b^{\beta} \leq \alpha a + \beta b$ if $\alpha+\beta = 1, a, b > 0$.

87. Harmonic Geometric Mean Inequality: Let $a, b > 0$ be real numbers. Then $\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab}$. Generalized inequality: $\frac{1}{\frac{\alpha}{a}+\frac{\beta}{b}} \leq a^{\alpha}b^{\beta}$ when $\alpha+\beta=1$ and $a, b > 0$.

88. n-term Arithmetic Geometric mean inequality: Let $\alpha_1, \ldots, \alpha_n > 0$ with $\sum_{j=1}^{n}\alpha_j = 1$. Then $a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n} \leq \sum_{j=1}^{n}\alpha_j a_j$.

89. Discrete Holder inequality: Let $p, q > 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $a_1, \ldots a_n, b_1, \ldots, b_n > 0$ be real numbers. Then:

$$\sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}$$

90. Holder Inequality: Let $p, q > 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let f, g be non-negative intetrable functions on an interval $[a,b]$. Then:

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)^q dx\right)^{\frac{1}{q}}$$

91. Jensen's Inequality: Let g be a convex function and f be a non-negative integrable function on an interval $[a,b]$. Then:

$$g\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \leq \frac{1}{b-a}\int_a^b g(f(x))dx$$

92. Dot product: $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

93. Cross product: $\vec{a}\times\vec{b} = a_1b_2 - b_1a_2$. $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta$.

94. Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

95. Roots of unity: $e^{\frac{2\pi i j}{k}}$ for $j = 0, 1, \ldots, k-1$.

96. Fundamental theorem of algebra: Let $p(z)$ be a polynomial with complex coefficients and degree $k \geq 1$. Then there is a complex number z with $p(z)=0$.

97. Complex logarithm: $\log(re^{i\theta}) = \log r + i\theta$. Note that there is ambiguity in the selection of θ.

98. Complex sine: $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

99. **Analytic Function:** A complex valued function of a complex variable $f(z)$ which is differentiable at z as a function of two variables is analytic at z if $df(z)$ is a complex multiple of dz. Let $f(z) = u(x,y) + iv(x,y)$. Then the required conditions are that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$.

100. Order of partial differentiation: Let F be a function of two variables whose first partial derivatives have continuous first partial derivatives. Then:

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

101. **Cauchy's Theorem** Let f be analytic with continuous derivative on a rectangle R. Let α be a closed curve lying in rectangle R. Then $\oint_\alpha f(z)dz = 0$.

102. Integration of $\frac{1}{z}$: $\oint_\alpha \frac{dz}{z} = 2\pi i$.