ACM95a Own Notes (Brown and Churchill), 6th ed.

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Chapter 1

Definitions

1. **Analytic Function:** A function f is analytic in an open set if it has a derivative at each point in that set. Also call f regular or holomorphic. f is analytic at a point z_0 if it is analytic in a neighbourhood of z_0.

2. **Entire Function:** An entire function is a function that is analytic at each point in the entire finite plane. Every polynomial is an entire function.

3. **Singular Point:** If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is a singular point or singularity of f.

4. **Isolated singular point:** A singular point z_0 is isolated if there is a deleted neighbourhood $0 < |z - z_0| < \epsilon$ of z_0 throughout which f is analytic.

5. **Harmonic conjugate:** If u and v are harmonic in D and their first-order partial derivatives satisfy the Cauchy-Riemann equations throughout D, then v is a harmonic conjugate of u.

6. **Domain vs Region** A domain is an open set (all points interior points) that is connected. A region is a domain together with some (or none or all) of its boundary points.

7. **Accumulation point** A point $z_0 \in S$ is an accumulation point of set S if each deleted neighbourhood of z_0 contains at least one point of S. If a set S is closed, then it contains each of its accumulation points.

8. **Multi-valued logarithm function:** $logz = ln|z| + i\arg(z)$, where $\arg(z)$ is the multi-valued argument function.

9. **Branch:** A branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values $f(z)$.

10. **Branch cut:** A branch cut is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Points on the branch cut for F are singular points of F and any point that is common to all branch cuts of f is called a **branch point**.

11. **Principal value:** The principal value of the multivalued function $f(z)$ can be found using $e^{Log(f(z))}$ where Log is the single-valued logarithm function.

12. **Absolute convergence:** A series of complex numbers $z_n = x_n + iy_n, n = 1, 2, \ldots$ converges absolutely if the series of real numbers $\sum_{n=1}^{\infty} |z_n|$ converges. Note that absolute convergence of a series of complex numbers implies convergence of that series.

13. **Circle of convergence:** The largest circle centred at z_0 such that the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for every point inside is the circle of convergence of the series. The series cannot converge at any point z_2 outside the circle of convergence.

14. **Uniform Convergence:** (Stronger condition than pointwise convergence) A sequence of functions $\{f_n\}$ converges uniformly to a limiting function f if the speed of convergence of $f_n(x)$ to $f(x)$ does not depend on x. More rigorously, if for all $\epsilon > 0$ there exists a natural number N_ϵ such that $|f_n(z) - f(z)| < \epsilon$ whenever $N > N_\epsilon$ and N_ϵ depends only on the value of ϵ and is independent of the point z taken in a specified region within the circle of convergence, then the convergence is said to be uniform in the region.

15. **Analytic Continuation:** Consider two domains D_1 and D_2 with non-empty intersection $D_1 \cap D_2$. Let a function f_1 be analytic in D_1. If a function f_2 is analytic in D_2 and has $f_2(z) = f_1(z)$ for $z \in D_1 \cap D_2$ then f_2 is an analytic continuation of f_1 into the second domain D_2.

16. **Residue:** Let C be a positively oriented simple closed curve around the isolated singular point z_0 and lying in the punctured disk $0 < |z - z_0| < R_2$, where R_2 represents the radius of convergence of the function f around z_0. Then the complex number $\frac{1}{2\pi i} \int_C f(z)dz$ is called the residue of f at the isolated singular point z_0. Note that the residue is the coefficient of $\frac{1}{z-z_0}$ in the Laurent series. If you can somehow figure out the Laurent series of the function around z_0 without explicit integration, you immediately know the residue.

17. **Pole:** If the principal part (negative powers of $z - z_0$) of f at isolated singular point z_0 contains at least one non-zero term but with a finite number of terms such that there exists a positive integer m with $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \cdots = 0$, then the isolated singular point z_0 is called a pole of order m. A pole of order 1 is called a simple pole.

18. **Removable singular point:** If the principal part of f at isolated singular point z_0 are all zero, then z_0 is known as a removable singular point. The residue at a removable singular point is always zero (of course, since the residue is the coefficient of the $n = -1$ term). The removable singular point can be removed by redefining f to have the value a_0, the first term in the Laurent series.

19. **Essential singular point:** If the principal part of f at isolated singular point z_0 contains an infinite number of non-zero terms, z_0 is said to be an essential singular point of f.

20. **Zeros of order m:** Consider a function f that is analytic at z_0. If $f(z_0) = 0$ and there is a positive integer m such that $f^{(m)}(z_0) \neq 0$ and each derivative of lower order at z_0 vanishes, then f is said to have a zero of order m at z_0.

Chapter 2

Theorems

1. **Necessary Conditions for Differentiability:** Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f'(z)$ exists at $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v exist there and satisfy the **Cauchy-Riemann Equations** $u_x = v_y, u_y = -v_x$. Write $f'(z_0) = u_x + iv_x$.

2. **Sufficient Conditions for Differentiability:** Suppose $f(z) = u(x,y) + iv(x,y)$ is defined in the ϵ-neighbourhood of $z_0 = x_0 + iy_0$. Suppose that the first-order partial derivatives of u and v exist everywhere in that neighbourhood and that they are continuous at (x_0, y_0). Then if the partial derivatives satisfy the **Cauchy-Riemann Equations** $u_x = v_x, u_y = -v_y$ at z_0, then the derivative $f'(z_0)$ exists.

3. **Sufficient condition for Differentiability (Polar Coordinates):** Given $f(z) = u(r,\theta) + iv(r,\theta)$ that is defined in some ϵ-neighbourhood of a non-zero point $z_0 = r_0 \exp(i\theta_0)$. Suppose that the first order partial derivatives of u and v exist everywhere in that neighbourhood and are continuous at (r_0, θ_0). Then if the partial derivatives satisfy the **Polar Form of the Cauchy-Riemann Equations** $u_r = \frac{1}{r}v_\theta, \frac{1}{r}u_\theta = -v_r$ at (r_0, θ_0), then the derivative $f'(z_0)$ exists. Write $f'(z_0) = e^{-i\theta}(u_r + iv_r)$.

4. **Reflection Principle:** Suppose that a function f is analytic in some domain D which contains a segment of the x-axis and is symmetric to that axis. Then $\overline{f(z)} = f(\bar{z})$ for each point $z \in D$ iff $f(x)$ is real for each point x on the segment.

5. **Harmonic Function:** A real-valued function $H : \mathbb{R}^2 \to \mathbb{R}$ is harmonic in a given domain of the xy plane if it has continuous partial derivatives of the first and second order throughout that domain that satisfies the partial differential equation $H_{xx}(x,y) + H_{yy}(x,y) = 0$, known as Laplace's equation.

6. **Analyticity implies components are harmonic:** If $f(z) = u(x,y) + iv(x,y)$ is analytic in D, then its component functions u and v are harmonic in D.

7. **Antiderivatives:** Suppose a function f is continuous on a domain D. Then the following are equivalent: (a) f has an antiderivative F in D, (b) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value; (c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

8. **Cauchy-Goursat Theorem:** If a function f is analytic at all points interior to and on a simple closed contour C, then $\int_C f(z)dz = 0$.

9. **Cauchy-Goursat for simply connected domain:** If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)dz = 0$ for every closed contour C lying in D. Note that C can intersect itself a finite number of times since we can divide C into a finite number of simple closed contours.

10. **Cauchy-Goursat for multiply connected domain:** Let C and C_k, $k = 1, 2, \ldots, n$ be simple closed curves in D. Let C be positively oriented. Let C_k be negatively oriented, pairwise disjoint, and interior to C. Then if f is analytic throughout the interior of C that is exterior to all C_k, then $\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$.

11. **Cauchy Integral Formula:** Let f be analytic everywhere within and on a simple closed positively oriented curve C. If z_0 is interior to C, then $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}$. The values of f interior to C are completely determined by the values of f on C.

12. **Analytic functions and higher-order derivatives:** If a function f is analytic at a point, then its derivatives of all orders are also analytic functions at that point. Also, its component functions u and v have continuous partial derivatives of all orders at that point.

13. **Higher-order derivatives of analytic functions:** $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}, n = 0, 1, 2, \ldots$.

14. **Sufficient condition for analyticity using closed integrals:** If a function f is continuous throughout a domain D, and if $\int_C f(z)dz = 0$ for every closed contour C lying in D, then f is analytic throughout D.

15. **Cauchy's inequality:** Let z_0 be a fixed complex number. If a function f is analytic within and on a circle $|z-z_0| = R$, taken in the positive orientation and denoted by C, and M_R is the maximum value of $|f(z)|$ on C, then $\left|f^{(n)}(z_0)\right| \leq \frac{n!M_R}{R^n}, n = 1, 2, \ldots$.

16. **Liouville's Theorem:** If f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane. The only entire function that is bounded in the complex plane is a constant. This follows from Cauchy's inequality when $n = 1$ and observing that the inequality must hold for arbitrarily large values of R.

17. **Fundamental Theorem of Algebra:** Any polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n, a_n \neq 0$ of degree $n \geq 1$ has at least one zero. That is, $\exists$ at least one point z_0 such that $P(z_0) = 0$.

18. **Maximum Modulus Principle:** If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$.

19. **Corollary of the Maximum Modulus Principle:** Suppose that a function f is continuous in a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

20. **Summation of series:** Suppose that $z_n = x_n + iy_n, n = 1, 2, \ldots$ and $S = X + iY$. Then $\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

21. **Taylor Series:** Suppose a function f is analytic throughout an open disk $|z - z_0| < R_0$. Then at each point z in that disk, $f(z)$ has the series representation $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$.

22. **Laurent's Theorem and Series:** Suppose that a function f is analytic throughout an annular domain $R_1 < |z-z_0| < R_2$ and let C denote any positively oriented simple closed contour around z_0 and lying in that annular domain. Then at each point z in the domain, $f(z)$ has the series representation $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, n = 0, 1, 2, \ldots$ and $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}, n = 1, 2, \ldots$. The series in the second term involving negative powers of $z - z_0$ is called the principal part of f at z_0. Alternatively, write the expansion as $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$ with $c_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, n = 0, \pm 1, \pm 2, \ldots$. Note that when f is analytic, then all the $b_n = 0$ and the expansion becomes the Taylor series.

23. **Determining absolute convergence:** If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges when $z = z_1, z_1 \neq z_0$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ with $R_1 = |z_1 - z_0|$.

24. **Uniform convergence within circle of convergence:** If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, then that series is uniformly convergent in the closed disk $|z - z_0| \leq R_1$ where $R_1 = |z_1 - z_0|$.

25. **Corollary of uniform convergence theorem:** A power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$. Also, if the power series $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges at a point $z_1 \neq z_0$ then it must converge absolutely to a continuous function in the domain exterior to the circle $|z-z_0| = R_1$ where $R_1 = |z_1 - z_0|$. Also, if a Laurent series representation $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is valid in an annulus $R_1 < |z - z_0| < R_2$ then both of the series converge uniformly in any closed annulus which is concentric to and interior to that region of validity.

26. **Integration of power series:** Let C denote any contour interior to the circle of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and let $g(z)$ be any function continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; that is $\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz$. Note that if $g(z) = 1$ over the domain then $\int_C S(z)dz = 0$ for every closed contour C.

27. **Series sum is analytic in circle of convergence:** The power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is analytic at each point z interior to the circle of convergence of that series.

28. **Differentiation of power series:** The power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ can be differentiated term by term in the interior of its circle of convergence to obtain $S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$.

29. **Uniqueness of Taylor series representations:** If a series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$ then it is the Taylor series expansion for f in powers of $z - z_0$.

30. **Uniqueness of Laurent series representations:** If a series $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

31. **Cauchy's Residue Theorem:** Let C be a positively oriented simple closed curve. If a function f is analytic inside and on C except for a finite number of singular points $z_k, k = 1, 2, \ldots$ inside C then $\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$.

32. **Alternative method to calculate the Residue Theorem:** If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed curve C, then $\int_C f(z)dz = 2\pi i \text{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$. Instead of calculating the residues are multiple locations, we only need to calculate one residue of a related function.

33. **Theorem regarding poles and the original function:** An isolated singular point z_0 of a function f is a pole of order m if and only iff $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0. Moreover, $\text{Res}_{z=z_0} f(z) = \phi(z_0)$ if $m = 1$ and $\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 2$.

34. **Lemma regarding zeros of order m:** A function f that is analytic at a point z_0 has a zero of order m there iff there is a function g which is analytic and non-zero at z_0 such that $f(z) = (z - z_0)^m g(z)$.

35. **Zeros can generate poles:** Let functions p and q be analytic at z_0 with $p(z_0) \neq 0$. Then if q has a zero of order m at z_0, then the quotient $p(z)/q(z)$ has a pole of order m there.

36. **Finding residues of simple poles:** Let two functions p and q be analytic at point z_0. If $p(z_0) \neq 0, q(z_0) = 0, q'(z_0) \neq 0$, then z_0 is a simple pole of the quotient $p(z)/q(z)$ and $\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

37. **Zero over a domain:** If a function f is analytic throughout a domain D and $f(z) = 0$ at each point z of a domain or arc contained in D, then $f(z) = 0$ in D.

38. **Unique determination of analytic function:** A function that is analytic in a domain D is uniquely determined over D by its values over a domain, or along an arc, contained in D.

39. **Riemann's theorem:** Suppose that a function f is analytic and bounded in some deleted neighbourhood $0 < |z - z_0| < \epsilon$ of a point z_0. If f is not analytic at z_0, then it has a removable singularity there.

40. **Casorati-Weierstrass Theorem:** Suppose that z_0 is an essential singularity of a function f, and let w_0 be any complex number. Then, for any $\epsilon > 0$, the inequality $|f(z) - w_0| < \epsilon$ is satisfied at some point z in each deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0.

Chapter 3

Syllabus-specific Notes

3.1 Part 1

1. **Introduction to complex numbers**
   - Definition of sum $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.
   - Definition of product: $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2)$.
   - Algebraic properties: Commutative ($z_1 + z_2 = z_2 + z_1, z_1z_2 = z_2z_1$), Associative ($(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $(z_1z_2)z_3 = z_1(z_2z_3)$).
   - Multiplicative inverse of $z = (x, y)$ is $z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ for $z \neq 0$.
   - Binomial formula: $(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   - Triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$.
   - Modified triangular inequality: $|z_1 \pm z_2| \geq ||z_1| - |z_2||$. Proof: Write $|z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2|$ so $|z_1 + z_2| \geq |z_1| - |z_2|$ when $|z_1| \geq |z_2|$. If $|z_1| < |z_2|$, interchange z_1 and z_2 to get $|z_1 + z_2| \geq -(|z_1| - |z_2|)$.
   - Conjugate of sum is sum of conjugates $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
   - Conjugate of product is product of conjugates $\overline{z_1z_2} = \overline{z_1} \cdot \overline{z_2}$.
   - Parabola: Consider a line (directrix) and a point (focus). The parabola is the locus of points equidistant to the directrix and the focus.
   - Hyperbola: A hyperbola is the locus of points whose absolute value of the difference of distances to two foci is a constant (2a). 2a is also the distance between its vertices. The canonical equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
   - Partial fraction decomposition: Consider a denominator with $(z - a_1)^{m_1}(z - a_2)^{m_2} \cdots$, where a_j are complex numbers and m_j are multiplicities. Then the partial fraction decomposition has denominators $(z - a_1)^{m_1}, (z - a_1)^{m_1 - 1}, \cdots, (z - a_1), (z - a_2)^{m_2}, \cdots (z - a_2)$.
   - Homework 1 Identity: if $|\alpha| < 1$ and $|\beta| < 1$, then $\left|\frac{\alpha-\beta}{1-\alpha\beta}\right| < 1$.
2. **Polar form**
   - Principal value Arg is the unique value Θ such that $-\pi < \Theta \leq \pi$. $\arg z = \text{Arg} z + 2n\pi$.
   - Inversion of $re^{i\theta}$ is $\frac{1}{r}e^{-i\theta}$.
   - Argument of product: $\arg(z_1z_2) = \arg z_1 + \arg z_2 + 2\pi k, k \in \mathbb{Z}$. Note that this may not work when you use Arg.
3. **Euler's formula** $e^{i\theta} = \cos\theta + i\sin\theta$.
4. **Complex exponential** The transformation $w = e^z$ maps the rectangular region $a \leq x \leq b, c \leq y \leq d$ onto the region $e^a \leq \rho \leq e^b, c \leq \phi \leq d$. The mapping is one-to-one if $d - c < 2\pi$.
5. **Trigonometric functions**
   $\sin\frac{\pi}{6} = \frac{1}{2}, \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \sin\frac{\pi}{2} = 1$.
   $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \cos\frac{\pi}{3} = \frac{1}{2}, \cos\frac{\pi}{2} = 0$.
   $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, \tan\frac{\pi}{4} = 1, \tan\frac{\pi}{3} = \sqrt{3}, \tan\frac{\pi}{2} = \infty$.
   More definitions and identities:

- $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
- $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
- $2 \sin z_1 \cos z_2 = \sin(z_1 + z_2) + \sin(z_1 - z_2)$
- $\sin(z + \pi/2) = \cos z$
- $\sin(z - \pi/2) = -\cos z$
- $\sin(iy) = i \sinh y, \sin z = -i \sinh(iz), \sinh z = -i \sin(iz)$
- $\cos(iz) = \cosh z, \cosh(iz) = \cos z$
- $\sin z = \sin x \cosh y + i \cos x \sinh y$
- $\cos z = \cos x \cosh y - i \sin x \sinh y$
- $|\sin z|^2 = \sin^2 x + \sinh^2 y, |\cos z|^2 = \cos^2 x + \sinh^2 y$
- $\sin z = 0 \iff z = n\pi, n \in \mathbb{Z}$
- $\cos z = 0 \iff z = \pi/2 + n\pi, n \in \mathbb{Z}$
- $\frac{d}{dz} \tan z = \sec^2 z$, $\frac{d}{dz} \cot z = -\csc^2 z$, $\frac{d}{dz} \sec z = \sec z \tan z$, $\frac{d}{dz} \csc z = -\csc z \cot z$
- $\cosh^2 y - \sinh^2 y = 1$
- $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$
- $\sinh z = \sinh x \cos y + i \cosh x \sin y, \cosh z = \cosh x \cos y + i \sinh x \sin y$
- $|\sinh z|^2 = \sinh^2 x + \sin^2 y, |\cosh z|^2 = \sinh^2 x + \cos^2 y$
- $\frac{d}{dz} \tanh z = sech^2 z$, $\frac{d}{dz} \coth z = -csch^2 z$, $\frac{d}{dz} sechz = -sechz \tanh z$, $\frac{d}{dz} cschz = -cschz \coth z$
- Quadrant-specific argument (multiple conditions hold for overlapping areas):

$$Arg(z) = \begin{cases} -Tan^{-1}(x/y) - \pi/2, & y < 0 \\ Tan^{-1}(y/x), & x > 0 \\ -Tan^{-1}(x/y) + \pi/2 & , y > 0 \end{cases}$$

  where $Tan^{-1} t$ has a unique solution on the interval $(-\pi/2, \pi/2)$.

6. **deMoivre's formula** $(cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$.

7. **Integer powers and roots** nth roots $\sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], k = 0, 1, \ldots, n-1$. Roots of unity $\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right), k = 0, 1, \ldots, n-1$.

8. **Complex logarithm** $\log z = \ln r + i(\theta + 2n\pi), n \in \mathbb{Z} = \ln|z| + i \arg z$. Identities: $\log(z_1 z_2) = \log z_1 + \log z_2, \arg(z_1 z_2) = \arg z_1 + \arg z_2$. This does not work for the principal branch.

9. Multiple-valuedness

10. Periodicity e^z is periodic with pure imaginary period $2\pi i$.

11. **Complex exponents** When $z \neq 0$ and $c \in \mathbb{C}$, then $z^c \equiv e^{c \log z}$. We also have $\frac{d}{dz} z^c = cz^{c-1}, |z| > 0, \alpha < \arg z < \alpha + 2\pi$. We also define $c^z = e^{z \log c}$, $\frac{d}{dz} c^z = c^z \log c$.

12. **Inverse trigonometric functions**

- $\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]$
- $\cos^{-1} z = -i \log[z + i(1 - z^2)^{1/2}]$, branch points at ± 1
- $\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z} = \frac{1}{2i} \log \frac{1+iz}{1-iz}$, branch points at $\pm i$.
- $\frac{d}{dz} \sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}$, depends on what branch square root is defined on. Branch points at ± 1.
- $\frac{d}{dz} \cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}$, depends on what branch square root is defined on. Branch points at ± 1
- $\frac{d}{dz} \tan^{-1} z = \frac{1}{1+z^2}$, branch points at $\pm i$
- $\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}]$, branch points at $\pm i$
- $\cosh^{-1} z = \log[z + (z^2 - 1)^{1/2}]$
- $\tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}$, branch points at ± 1.
- $\coth^{-1} z = \frac{1}{2} \log \frac{z+1}{z-1}$

- $csch^{-1}z = \log\left(\frac{1}{z} + (\frac{1}{z^2}+1)^{1/2}\right)$
- $sech^{-1}z = \log\left(\frac{1}{z} + (\frac{1}{z^2}-1)^{1/2}\right)$

13. **One-to-one mappings** A function is one-to-one on a set S if the equation $f(z_1) = f(z_2)$ implies that $z_1 = z_2$.

14. Riemann surfaces

15. **Point at infinity:** Theorems involving limits:
    - $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0$
    - $\lim_{z\to\infty} f(z) = w_0 \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$
    - $\lim_{z\to\infty} f(z) = \infty \iff \lim_{z\to 0} \frac{1}{f(1/z)} = 0$

16. **Stereographic projection**
    - If $Z = (x_1, x_2, x_3)$ is the projection on the Riemann sphere of the point $z = x + iy$ in the complex plane, then $x_1 = \frac{2\Re(z)}{|z|^2+1}, x_2 = \frac{2\Im(z)}{|z|^2+1}, x_3 = \frac{|z|^2-1}{|z|^2+1}$. This is because we can parametric the line through $(0, 0, 1)$, the North Pole, and $(x, y, 0)$ on the complex plane using $(x_1, x_2, x_3) = (tx, ty, 1-t), t \in (-\infty, \infty)$. Then we need to satisfy $x_1^2 + x_2^2 + x_3^2 = 1$. Find that $t = \frac{2}{1+|z|^2}$ to satisfy the condition.
    - Given (x_1, x_2, x_3) on the Riemann sphere, the complex plane values are $x = \frac{x_1}{1-x_3}, y = \frac{x_2}{1-x_3}$.
    - Lines and circles on the xy complex plane map to circles on the Riemann sphere. General equation for circle or line: $A(x^2 + y^2) + Cx + Dy + E = 0$. $A = 0$ for a line. To prove this, substitute the previous expressions for x and y using Riemann sphere coordinates into the general equation for a circle or line, then show that $Cx_1 + Dx_2 + (A - E)x_3 + A + E = 0$, which is the equation of a plane. The intersection of a plane and the Riemann sphere is clearly a circle.

17. **Branch points and branch cuts** A branch of a multi-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values $f(z)$. A branch cut is a portion of a line or curve that is introduced in order to define branch F of a multivalued function f. Points on the branch cut for F are singular points of F. Points that are common to all branch cuts of f are called branch points.

18. Branch point at infinity

19. **Regions of the complex plane**
    - An open set that is connected is a domain.
    - Any neighbourhood is a domain.
    - A domain together with some or none or all of its boundary points is a region.
    - A set S is bounded if every point of S lies inside some circle $|z| = R$.
    - A simply connected domain is such that every simple closed contour within it encloses only points of the domain.

20. **Limits and continuity** Let a function f be defined at all points z in some deleted neighbourhood of z_0 the limit of $f(z)$ as z approaches z_0 can be written as $\lim_{z\to z_0} f(z) = w_0$. This means that $\forall\epsilon > 0, \exists\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$. When a limit of a function exists, it is unique.
    - Complex limit is the sum of real limits: $\lim_{z\to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0, \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$ for $f(z) = u(x, y) + iv(x, y), w_0 = u_0 + iv_0$.
    - Sum and product of limits: Suppose that $\lim_{z\to z_0} f(z) = w_0, \lim_{z\to z_0} F(z) = W_0$. Then $\lim_{z\to z_0}[f(z) + F(z)] = w_0 + W_0, \lim_{z\to z_0}[f(z)F(z)] = w_0W_0, \lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}, W_0 \neq 0$

    A function f is continuous at a pont z_0 if:
    - $\lim_{z\to z_0} f(z)$ exists
    - $f(z_0)$ exists
    - $\lim_{z\to z_0} f(z) = f(z_0)$

    More theorems:
    - If two functions are continuous at a point, their sum and product are also continuous at that point, and their quotient is continuous at any point where the denominator is not zero.
    - A composition of continuous functions is itself continuous.

- If a function $f(z)$ is continuous and non-zero at a point z_0, then $f(z) \neq 0$ throughout some neighbourhood of that point.
- A function $f(z) = u(x, y) + iv(x, y)$ is continuous at a point (x_0, y_0) iff its component functions are continuous there.

21. **Complex derivative** Let f be a function whose domain of definition contains a neighbourhood of a point z_0. The derivative is $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_)}{z - z_0}$, for all $\arg z$ provided it exists. A function is differentiable at z_0 when its derivative at z_0 exists.

    - The existence of a derivative of a function at a point implies the continuity of the function at that point.
    - $\frac{d}{dz} \log z = \frac{1}{z}, |z| > 0, \alpha < \arg z < \alpha + 2\pi$, $\frac{d}{dz} Logz = \frac{1}{z}, |z|, -\pi < Argz < \pi$.

22. **Analyticity**

    - A function f is analytic in an open set if it has a derivative at each point in that set. Note that if we should speak of a function f that is analytic in a set S which is not open, then we are speaking that f is analytic in an open set containing S.
    - f is analytic at a point of it is analytic throughout some neighbourhood of z_0. (This means that $f(z) = |z|^2$ is not analytic at any point because its derivative only exists at $z = 0$).
    - An entire function is analytic at each point in the entire finite plane.
    - A singular point is a point z_0 such that f fails to be analytic at but with f analytic at some point in every neighbourhood of z_0.
    - A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D.

23. **Cauchy-Riemann equations**

    - Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v must exist at (x_0, y_0) and satisfy $u_x = v_y, u_y = -v_x$. Then we can write $f'(z_0) = u_x + iv_x$. These conditions are sufficient of the partial derivatives are continuous at (x_0, y_0).
    - In polar coordinates, write $z = re^{i\theta} = x + iy$ so that we can write $ru_r = v_\theta, u_\theta = -rv_r$ for the CR equations in polar coordinates. Write $f'(z_0) = e^{-i\theta}(u_r + iv_r)$.

24. **Complex integration** Existence of integral: Functions must be piecewise continuous - continuous everywhere except for a finite number of points with one-sided limits.

    - Fundamental Theorem of Calculus: Suppose the functions $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ are continuous on the interval $a \leq t \leq b$. If $W'(t) = w(t)$ when $a \leq t \leq b$, then $U'(t) = u(t)$ and $V'(t) = v(t)$. Also, $\int_a^b w(t)dw = U(b) - U(a) + iV(b) - iV(a)$.
    - Integral bounds: $\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt$.
    - A partition P_n of smooth curve γ is a finite number of points $z_0, z_1, \ldots, z_n$ on γ such that $z_0 = \gamma(a), z_n = \gamma(b)$.
    - The Riemann sum for the function f corresponding to partition P_n is given by $S(P_n) = \sum_{k=1}^n f(c_k)(z_k - z_{k-1})$, where c_k lies on the arc from z_{k-1} to z_k.
    - f is integrable along smooth curve γ if there exists a complex number L that is the limit of every sequence of Riemann sums $\{S(P_j)\}$ corresponding to any sequence of partitions of γ satisfying $\lim_{n \to \infty} \mu(P_n) = 0$, where $\mu(P_n)$ is the mesh of the partition, the largest of the lengths between consecutive points along the partition.
    - If f is continuous on the directed smooth curve γ, then f is integrable.

25. **Parameterization** A parametrisation of a smooth arc in the complex plane is the complex-valued function that takes real input $z(t), a \leq t \leq b$ that has a continuous derivative with respect to t on $[a, b]$, with $z'(t) \neq 0$ on $[a, b]$, and that is one-to-one on $[a, b]$.

    - If f is continuous on the directed smooth curve γ and if $z = z_1(t), t \in [a, b]$ and $z = z_2(t), t \in [c, d]$ are two admissible parametrizations with the same orientation, then $\int_a^b f(z_1(t))z_1'(t)dt = \int_c^d f(z_2(t))z_2'(t)dt$.
    - The parametrisation of a line passing through two points is clearly $z = (1 - t)z_1 + tz_2, t \in [0, 1]$.

26. **Contours**

- Arc: a set of points $z = (x, y)$ in the complex plane such that $x = x(t), y = y(t), a \leq t \leq b$ where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. This is a continuous mapping of the interval $a \leq t \leq b$ onto the z-plane.
- Simple/Jordan arc: Does not cross itself: $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$.
- Closed: $z(b) = z(a)$.
- Change of variables: Let $t = \phi(\tau), \alpha \leq \tau \leq \beta$ where ϕ is a real-valued function mapping $\alpha \leq \tau \leq \beta$ onto the interval $a \leq t \leq b$. Assume that $\phi'(\tau) > 0, \forall \tau$. Then $z = Z(\tau), \alpha \leq \tau \leq \beta$, where $Z(\tau) = z(\phi(\tau))$.
- Contour: Piecewise smooth arc. Consist of a finite number of smooth arcs joined end to end.
- Define a positive orientation to be such that the interior lies to the left when the curve is traced.
- The length of a contour parametrised by $z(t) = x(t) + iy(t), a \leq t \leq b$ is given by $\int_a^b \frac{ds}{dt} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$.

27. **Fundamental theorem for contour integration**

- Contour integration: $\int_C f(z) dz = \int_a^b f[z(t)] z'(t) dt$ for $f(z)$ piecewise continuous on C, a contour parametrized by $z(t), a \leq t \leq b$. This value is invariant under a change in representation.
- Fundamental Theorem: Suppose a function $f(z)$ is continuous in domain D and has an antiderivative $F(z)$ throughout D (i.e. $F'(z) = f(z), z \in D$). The for any contour Γ lying in D, with initial point z_I and terminal point z_T, we have $\int_\Gamma f(z) dz = F(z_T) - F(z_I)$. Note that the conditions of theorem imply that $F(z)$ is analytic (since it has a derivative $f(z)$) and hence is continuous on D.
- Upper bound: $\left|\int_C f(z) dz\right| \leq M \int_a^b |z'(t)| dt$ where $|f(z)| \leq M$. The RHS integral is the length of the contour. Hence we write $\left|\int_C f(z) dz\right| \leq ML$
- Upper bound using moduli: If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \leq t \leq b$, then $\left|\int_a^b w(t) dt\right| \leq \int_a^b |w(t)| dt$.
- Using bounds: Bound the numerator from above using the triangle inequality $|a + b| \leq |a| + |b|$. Bound the denominator from below using the modified triangle inequality $|a + b| \geq ||a| - |b||$

28. **Equivalence between existence of antiderivatives, Vanishing of closed contour integrals and independence of path**

- Theorem: Existence of antiderivative. Suppose a function $f(z)$ is continuous in a domain D. The following are equivalent:
  - $f(z)$ has an antiderivative $F(z)$ throughout D.
  - The integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value: $\int_{z_1}^{z_2} f(z) dz = F(z)|_{z_1}^{z_2} = F(z_2) - F(z_1)$. The contour integrals are path-independent.
  - The integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

29. **Cauchy-Goursat theorem/Cauchy's Integral Theorem**

- Green's Theorem: Suppose we have two real-valued functions $P(x, y)$ and $Q(x, y)$, which have continuous first-order partial derivatives throughout the closed region R made of all the points interior and on the simple closed contour C. Then $\int_C P dx + Q dy = \iint_R (Q_x - P_y) dA$.
- Statement of Theorem: If a function f is analytic at all points interior to and on a simple closed contour C, then $\oint_C f(z) dz = 0$.

30. **Extensions to self-intersecting contours and multiply-connected domains** Suppose that C is a simple closed contour, described counterclockwise. Let $C_k, k = 1, 2, \ldots, n$ be simple closed contours interior to C, all described in the clockwise direction and mutually disjoint. If f is analytic on all of these contours and throughout the multiply connected domain consisting of points interior to C and exterior to each C_k, then $\int_C f(z) dz + \sum_{k=1}^n \int_{C_k} f(z) dz = 0$. Directions have been chosen such that the multiply connected domain lies to the left of the path.

31. **Deformation of contours (Principle of deformation of paths)**

- Let C_1 and C_2 denote positively-oriented simple closed contours, C_1 interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between then, then $\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$.
- A loop Γ_0 is said to be continuously deformable to the loop Γ_1 in the domain D if there exists a function $z(s, t)$ continuous on the unit square $0 \leq s \leq 1, 0 \leq t \leq t$ that satisfies the following conditions:
  - For each fixed $s \in [0, 1]$, the function $z(s, t)$ parametrises a loop lying in D.

  - The function $z(0,t)$ parametrises the loop Γ_0.
  - The function $z(1,t)$ parametrises the loop Γ_1.

- Deformation Invariance Theorem: Let f be an analytic function in the domain D containing the loops Γ_0, Γ_1. If these loops can be continuously deformed into one another in D, then $\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz$.

32. **Cauchy integral formula** Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0}$.

    - Gauss Mean Value Theorem: When a function is analytic within and on a given circle (parametrised by $z = z_0 + \rho e^{i\theta}$), its value at the centre is the arithmetic mean of its values on the circle: $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta$.

33. **Derivatives of analytic functions** Let f be analytic inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, n = 0, 1, 2, \ldots$.

    Consequences:

    - If a function f is analytic at a given point, then its derivatives of all orders exist and are analytic there too.
    - If a function $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z = (x,y)$, then the component functions u, v have continuous partial derivatives of all orders at that point.
    - Let f be continuous on domain D. If $\int_C f(z)dz = 0$ for every closed contour C on D, then f is analytic throughout D.

34. **Generalized Cauchy integral formula**

35. **Morera's theorem** If $f(z)$ is continuous in a region R and satisfies $\oint_C f(z)dz = 0$ for all closed contours in R, then $f(z)$ is analytic in R. Note that the region is not required to be simply connected.

36. **Cauchy's inequality** Suppose a function f is analytic inside and on a positively oriented circle C_R, centred at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then $|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}, n = 1, 2, \ldots$.

    - Maximum modulus principle: Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighbourhood $|z - z_0| < \epsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighbourhood.
    - Alternative statement of Maximum Modulus Principle: If f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. The maximum value of $|f(z)|$ occurs somewhere on the boundary and never in the interior.

37. **Liouville's theorem**

    - Statement: If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.
    - Consequence: Fundamental Theorem of Algebra: Any polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n, a_n \neq 0$ of degree $n \geq 1$ has at least one zero.
    - If a function is analytic everywhere in the extended complex plane except for a pole at z_0, write $f(z) = \sum_{n=-m}^{-1} a_n(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^n$ where the first term is the principal part. Then the second term must be bounded and analytic everywhere since the Laurent series converges for all $z \neq z_0$. Hence the second term must be a constant.
    - If a function has exactly one pole at infinity of order m, then write $f(1/w) = \sum_{n=-m}^{-1} a_n w^n + \sum_{n=0}^{\infty} a_n w^n$. Since $f(z)$ is bounded near $z = 0$, $f(1/w)$ is bounded for large $|w|$. Hence the second term forms an analytic function that is bounded everywhere, and hence is a constant. Hence $f(z) = a_{-m}z^m + a_{-m+1}z^{m-1} + \ldots + a_{-1}z + a_0$ is a polynomial.

38. **Harmonic functions**

    - A harmonic function is a real-valued function of two variables with continuous first and second partial derivatives that satisfies $H_{xx} + H_{yy} = 0$. In the polar form $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$.
    - If ϕ is harmonic in a simply connected domain D and $\phi(z)$ achieves its maximum or minimum value at some point $z_0 \in D$, then ϕ is constant in D.
    - A harmonic function in a bounded simply connected domain attains its maximum and minimum on the boundary.
    - Let ϕ_1, ϕ_2 be harmonic in a bounded domain D. Suppose that $\phi_1 = \phi_2$ on the boundary of D. Then $\phi_1 = \phi_2$.

39. **Harmonic conjugates**

- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in D then u and v are harmonic in D.
- f is analytic in D iff v is a harmonic conjugate of u.
- When trying to simplify a function of x and y into $z = x + iy$ alone, make the substitutions $x = (z + \bar{z})/2, y = (z - \bar{z})/2i$ and solve for z. The analytic function should not contain $|z|$ or $\bar{z}$.
- The level curves of harmonic functions and their harmonic conjugates intersect at right angles. Note that the gradient vectors of the level curves (normal to the level curves) $\nabla u \cdot \nabla v = u_x v_x + u_y v_y$ vanishes by CR equations, hence they are orthogonal.
- **Existence of harmonic conjugate:** If a harmonic function $u(x, y)$ is defined on a simply-connected domain D it always has a harmonic conjugate $v(x, y)$ in D.

40. **Potential flow applications**

- Let the vector $V = p + iq$ denote the velocity of a particle of a fluid at point (x, y).
- The circulation of the fluid along any contour C is defined as $\int_C V_T(x, y)d\sigma$, where $V_T(x, y)$ is the tangential component of the velocity vector along C.
- The mean speed is the ratio of the circulation along C to the length of C.
- Using Green's Theorem, we can write $\int_C V_T(x, y)d\sigma = \iint_R [q_x(x, y) - p_y(x, y)]dA$
- The rotation of the fluid is defined by $\omega(x, y) = \frac{1}{2}[q_x(x, y) - p_y(x, y)]$. If $\omega(x, y) = 0$ at each point in a simply connected domain, the flow is called irrotational in that domain.
- Incompressible, viscosity-free fluids obey Bernoulli's equation: $\frac{P}{\rho} + \frac{1}{2}|V|^2 = c$, where c is a constant and $P(x, y)$ is the fluid pressure.
- For an irrotational flow in a simply connected domain, $p_y = q_x$, and hence $\phi(x, y) = \int_{(x_0,y_0)}^{(x,y)} p(s, t)ds + q(s, t)dt$ is independent of path. Call $\phi(x, y)$ the velocity potential, and note that $\phi_x = p, \phi_y = q$. The velocity potential must satisfy Laplace's equation in an incompressible fluid, and hence is a harmonic function.
- The velocity vector can be written as $V = \nabla\phi = \phi_x(x, y) + i\phi_y(x, y)$.
- Let $\psi(x, y)$ denote the harmonic conjugate of $\phi(x, y)$. The velocity vector is tangent to the curves $\psi(x, y) = c$, and these are called the streamlines of the flow. ψ is called the stream function.
- The analytic function $F(z) = \phi(x, y) + i\psi(x, y)$ is called the complex potential of the flow. The velocity can be written as $V = \overline{F'(z)}$ because $F'(z) = \phi(x, y) - i\phi_y(x, y)$ by the CR equations.
- Since $\psi(x, y) = \int_{(x_0,y_0)}^{(x,y)} -q(s, t)ds + p(s, t)dt$ by virtue of its being the harmonic conjugate of ϕ, we can write $\psi(x, y) = \int_C V_N(s, t)d\sigma$, where $V_N(x, y)$ is the normal component of the velocity vector. Hence it is the time rate of flow of the fluid across C.

3.2 Part 2

1. **Uniform convergence**

- Convergence of an infinite series: An infinite series $\sum_{n=1}^{\infty} z_n$ converges to the sum S if the sequence of partial sums $S_N = \sum_{n=1}^{N} z_n$ converges to S.
- The necessary condition for the convergence of a series is $\lim_{n\to\infty} z_n = 0$. The terms of a convergent series of complex numbers are hence bounded.
- Absolutely convergence of a series of complex numbers implies convergence of that series.
- Define the remainder $\rho_N = S - S_N$. A series converges to a number S if and only if $\rho_N \to 0$.
- Absolute convergence inside disk of convergence: If a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z = z_1, z_1 \neq z_0$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1, R_1 = |z_1 - z_0|$.
- Uniform convergence: $S_N(z)$ converges uniformly to $S(z)$ if for all $\epsilon > 0$, there is a positive integer N_ϵ such that $|S(z) - S_N(z)| < \epsilon$ whenever $N > N_\epsilon$ where N_ϵ depends only on the value of ϵ and is independent of the point z taken within the circle of convergence.

2. **Taylor series**

- Suppose a function f is analytic throughout a disk $|z - z_0| < R_0$. Then $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, where $a_n = \frac{f^{(n)}(z_0)}{n!}$. The Taylor series converges uniformly to $f(z)$ in any closed subdisk $|z - z_0| \leq R' < R$.

- Useful Taylor series:
  - $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
  - $\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}, |z| < \infty$
  - $\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}, |z| < \infty$
  - $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, |z| < \infty$. Substitute $z \to iz$ in sine expansion, then multiply by $-i$ since $\sinh z = -i \sin(iz)$.
  - $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, |z| < \infty$. Obtain from $\cosh z = \cos(iz)$.
  - $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1$.
  - $\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n (z-1)^n, |z-1| < 1$.
  - $\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \frac{1}{z^{n+1}} + \frac{1}{z^N} \frac{s^N}{z-s}$, note finite number of terms.
  - $\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$.
  - $\tan^{-1}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n-1}}{2n-1}$.

3. Uniqueness of analytic functions

4. **Power series**

   - Geometric sum: $\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$.

5. **Weierstrass M-test**

   - Motivation: Test for uniform convergence of an infinite series of functions.
   - Suppose that $\{f_n\}$ is a sequence of real or complex-valued functions defined on a set A, and that there is a sequence of positive numbers $\{M_n\}$ satisfying:

   $$\forall n \geq 1, \forall x \in A : |f_n(x)| \leq M_n, \sum_{n=1}^{\infty} M_n < \infty$$

   Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

6. **Circle of convergence**

   - **Limsup:** The limsup of a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is the smallest real number l with the property that for all $\epsilon > 0$ there are only a finite number of values of n such that x_n exceeds $l + \epsilon$. If no such number satisfies this property, set $\limsup x_n = \infty$. If all real numbers have this property, set $\limsup x_n = -\infty$.
   - The radius of convergence R of a sequence of coefficients $\{a_j\}$ is:

   $$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

   - If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then that series must be uniformly convergent in the closed disk $|z - z_0| \leq R_1$, where $R_1 = |z_1 - z_0|$.
   - A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a continuous function $S(z)$ inside its circle of convergence $|z-z_0| = R$.
   - Power series with negative powers: Write $w = \frac{1}{z-z_0}$. Then if the power series $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges at a point $z_1 \neq z_0$, the series $\sum_{n=1}^{\infty} b_n w_n = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges absolutely to a continuous function when $|w| < \frac{1}{|z_1-z_0|}$, which is the domain exterior to the circle $|z - z_0| = R_1 = |z_1 - z_0|$.
   - **Integral Test for convergence** For $f(x) \geq 0$ for $x \geq a$, let $I = \lim_{M\to\infty} \int_a^M f(x)dx$. Divergence and convergence of the infinite power series $\sum_{n=a}^{\infty} f(n)$ follows that of I.

7. Ratio test

8. **Integration of power series**

   - Let $g(z)$ be any function continuous on C, a contour interior to the circle of convergence of $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then the series multiplied by $g(z)$ can be integrated term-by-term over C:

   $$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz$$

9. **Analyticity of power series**

- The sum $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic at each point z interior to the circle of convergence of that series.

10. **Differentiation of power series**

- The power series $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be differentiated term-by-term at each point interior to its circle of convergence. That is,

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-z_0)^n$$

11. **Uniqueness of power series**

- If a series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z-z_0| = R$, then it is the Taylor series expansion for f in powers of $z-z_0$.
- To prove the above, write $g(z) = \frac{1}{2\pi i}\frac{1}{(z-z_0)^{n+1}}$ and evaluate $\int_C g(z)f(z)dz$ term-by-term. Prove and use the identity $\int_C (z-z_0)^{n-1}dz = \begin{cases} 0, n = \pm1, \pm2, \ldots \\ 2\pi i, n = 0 \end{cases}$ when C is a circle surrounding z_0. Then show that $\sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m dz = a_n = \frac{f^{(n)}(z_0)}{n!}$.
- If a series:

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z-z_0$ for that domain.

12. **Arithmetic operations on power series**

- Suppose each of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge within some circle $|z-z_0| = R$. The product of those sums has a Taylor series expansion $f(z)g(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}\frac{g^{(n-k)}(z_0)}{(n-k)!}$. This is called the **Cauchy product** of the two series.
- **Leibniz's Formula for nth derivative of fg:**

$$(fg)^{(j)} = \sum_{k=0}^{j} j! \frac{f^{(j-k)}}{(j-k)!}\frac{g^{(k)}}{k!} = \sum_{k=0}^{j} \binom{j}{k} f^{(j-k)} g^{(k)}$$

13. **Laurent series**

- Suppose a function f is analytic throughout an annular domain $R_1 < |z-z_0| < R_2$, and let C be any positively oriented simple closed contour around z_0 and lying in the annular domain. Then each point in the domain has the series representation:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$
$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$$

- Alternatively write the series as:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$
$$c_= \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, n = 0, \pm1, \pm2, \ldots$$

- If f is actually analytic throughout $|z-z_0| < R_2$ instead of just the annular domain, then all $b_n = 0$ because $\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}, n = 0, 1, 2, \ldots$ so the expansion is just the Taylor series.
- Examples:
  - $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}, 0 < |z| < \infty$.

14. **Zeros of analytic functions**
    - Suppose a function f is analytic at z_0. All the derivatives of f exist there. If $f(z_0) = 0$ and there is a positive integer m such that $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0$, then f has a zero of order m at z_0.
    - Let f be analytic at z_0. Then f has a zero of order m at z_0 iff there is a function g which is analytic and non-zero at z_0 such that $f(z) = (z-z_0)^m g(z)$.
    - Neighbourhood of zeroes: If f is analytic at z_0 and $f(z_0) = 0$ but $f(z)$ is not identically equal to zero (check that not all the derivatives of f at z_0 vanish) in any neighbourhood of z_0, then $f(z) \neq 0$ throughout some deleted neighbourhood $0 < |z-z_0| < \epsilon$ of z_0.
    - Zero along a line or domain: Given a function f and point z_0, suppose that f is analytic throughout a neighbourhood N_0 of z_0 and that $f(z) = 0$ at each point of a domain D or line segment L containing z_0. Then $f(z) \equiv 0$ throughout N_0.

15. **Isolated singularities**
    - Isolated singularity: A singular point z_0 is isolated if there is a deleted ϵ neighbourhood $0 < |z-z_0| < \epsilon$ of z_0 throughout which f is analytic.
    - Non-isolated singularity: A point where the function is not analytic in the deleted neighbourhood of that point. Examples: Branch point.
    - The zeros of polynomials in the denominator always result in isolated singular points because the zeros of a polynomial are finite in number (Fundamental Theorem of Algebra)
    - Isolated singular point at infinity: If there is a positive number R_1 such that f is analytic for $R_1 < |z| < \infty$ and f is singular at infinity then f has an isolated singular point at infinity.

16. **Removable singularities** The following statements are equivalent:
    - z_0 is a isolated removable singularity.
    - $|f(z)|$ is bounded and analytic in some deleted neighbourhood $0 < |z-z_0| < \epsilon$ of z_0 but not at z_0.
    - $f(z)$ has a finite limit as $z \to z_0$ - L'hopital's rule may be useful here.
    - $f(z)$ can be redefined at z_0 so that it is analytic at z_0. Write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, then define $f(z_0) = a_0$ so that the series represents an analytic function interior to its circle of convergence.
    - Every coefficient of the principal part of f is zero at z_0.
    - Riemann's theorem: Suppose a function f is bounded and analytic in some deleted neighbourhood $0 < |z-z_0| < \epsilon$ of z_0. If f is not analytic at z_0, then it has a removable singularity there.

17. **Poles** The following statements are equivalent:
    - $f(z)$ has a pole at z_0.
    - $|f(z)| \to \infty$ as $z \to z_0$.
    - $f(z) = \frac{g(z)}{(z-z_0)^n}, n \in \mathbb{Z}, n \geq 0$ and $g(z)$ is analytic at z_0, with $g(z_0) \neq 0$.
    - There exists a positive integer $m \geq 1$ such that $b_m \neq 0, b_{m+1} = b_{m+2} = \cdots = 0$. Then we write the Laurent series:

      $$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

      within its circle of convergence.
    - Zeros and Poles: Suppose that two functions p and q are analytic at z_0, $p(z_0) \neq 0$ and q has a zero of order m at z_0. Then $p(z)/q(z)$ has a pole of order m at z_0.

18. **Essential singularities** The following are equivalent:
    - z_0 is an isolated essential singularity.

- $|f(z)|$ is neither bounded nor goes to infinity for $z \to z_0$.
- $f(z)$ assumes every complex number except possibly one exception infinitely many times for every neighbourhood of z_0.
- An infinite number of the coefficients b_n in the principal part of f at z_0 are non-zero.
- Casorati-Weierstrass theorem: Suppose that z_0 is an essential singularity of f, and let ω_0 be any complex number. Then, for all $\epsilon > 0$, the inequality $|f(z) - \omega_0| < \epsilon$ is satisfied at some point z in each deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0

19. **Picard's theorem** In each neighbourhood of an essential singular point, a function assumes every finite value, except possibly one, an infinite number of times.

20. **Non-isolated essential singularities**

    - z_0 is an non-isolated essential singularity if many isolated singularities cluster around z_0 so there does not exist a deleted neighbourhood in which $f(z)$ is analytic. (Not the same behaviour as isolated essential singularity)

21. **Residues and calculating residues**

    - $\int_C f(z)dz = 2\pi i \text{Res}_{z=z_0} f(z)$ where C is any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |z - z_0| < R_2$ where f is analytic at each point z in the punctured disk.
    - Residue at infinity: Let f be analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C. Let R_1 be a positive number large enough such that C lies inside the circle $|z| = R_1$. then f is analytic throughout $R_1 < |z| < \infty$ and the point at infinity is an isolated singular point of f. The residue at infinity is defined to be $\int_{C_0} f(z)dz = 2\pi i \text{Res}_{z=\infty} f(z)$ where C_0 is the negatively oriented circle $|z| = R_0 > R_1$. Hence $\int_C f(z)dz = -2\pi i \text{Res}_{z=\infty} f(z)$.
    - Finding residue at infinity:

    $$\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} \left[\frac{1}{z^2} f(\frac{1}{z}) \right]$$

    this follows from the change of variables $\zeta = \frac{1}{z}$:

    $$\int_{|\zeta|=1/R} f\left(\frac{1}{\zeta}\right) \frac{d\zeta}{\zeta^2} = \int_{|z|=R} f(z)dz$$

    - Application of residue at infinity: If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a **positively oriented** simple closed contour C, then:

    $$\int_C f(z)dz = 2\pi i \text{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$$

    - Residues at poles: Let z_0 be an isolated singular point of f. Then z_0 is a pole of order m if $f(z)$ can be written as $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0. If $m = 1$, then $\text{Res}_{z=z_0} f(z) = \phi(z_0)$ and $\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ when $m = 2, 3, \ldots$.
    - Let two functions p and q be analytic at z_0 If $p(z_0) \neq 0, q(z_0) = 0, q'(z_0) \neq 0$, then z_0 is a simple pole of $p(z)/q(z)$ and $\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.
    - Residues of $\frac{1}{z^n+1}, n \in \mathbb{Z}, n \neq 1$. We note that the singularities are at the nth roots of -1, c_k. Note further that the singularities are simple poles because the numerator is $1 \neq 0$ and the derivative of the denominator nz^{n-1} does not vanish at each singularity. Then the residue at c_k is given by $\frac{1}{n(c_k)^{n-1}} = \frac{c_k}{n(c_k)^n} = \frac{-c_k}{n}$ because $(c_k)^n = -1$ since it is an nth root of -1.

22. **Cauchy's residue theorem**

    - Let C be a simple closed contour described positively. If a function f is analytic inside an on C except for a finite number of singular points $z_k, k = 1, 2, \ldots, n$ inside C then $\int_C f(z)dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$

23. **Real trigonometric integrals**

    - Given $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$, define $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z-1/z}{2i}, \cos\theta = \frac{z+1/z}{2}$.

- Identities:
  (a) $\sin(x + \pi/2) = \cos x$
  (b) $\sin(x - \pi/2) = -\cos x$
  (c) $\sin(x \pm \pi) = -\sin x$
  (d) $\cos(x + \pi/2) = -\sin x$
  (e) $\cos(x - \pi/2) = \sin x$
  (f) $\cos(x \pm \pi) = -\cos x$
  (g) $\tan(x + \pi) = \tan x$
  (h) $\tan(x \pm \pi/2) = -\cot x$
  (i) $\cos 3x = 4\cos^3 x - 3\cos x$
  (j) $\sin 3x = 3\sin x - 4\sin^3 x$
  (k) $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
  (l) $\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
  (m) $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$
  (n) $\cos a + \cos b = 2\cos\frac{a-b}{2}\cos\frac{a+b}{2}$

24. **Improper integrals**

- The improper integral of a continuous function $f(x)$ over the semi-infinite interval $0 \le x < \infty$ is defined by: $\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx$.

25. **Cauchy principal value**

- The Cauchy principal value of of $\int_{-\infty}^{\infty} f(x)dx$ is $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$.
- It is not always true that the infinite integral converges when its Cauchy principal value exists.
- Assume that $f(x), -\infty < x < \infty$ is an even function, and assume that the Cauchy principal value for the integral from $-\infty$ to ∞ exists. Then $\int_{-\infty}^{\infty} f(x)dx = P\int_{-\infty}^{\infty} f(x)dx$ and $\int_0^\infty f(x)dx = \frac{1}{2}P\int_{-\infty}^{\infty} f(x)dx$.
- Let $f(z) = \frac{p(z)}{q(z)}$ have a finite number of singularities, none of which lie in the real axis. Define the curve to be from $z = -R$ to $z = R$, and back to $z = -R$ through a semicircle of radius R. Then if $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$, where C_R is the semi-circular part (prove this by examining the bounds on integrals) then it follows that $P\int_{-\infty}^{\infty} f(z)dz = 2\pi i\sum_{k=1}^{n}\text{Res}_{z=z_k} f(z)$ where z_k are the singularities contained in the curve. If $f(x)$ is even, use the identity above to write $\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum_{k=1}^{n}\text{Res}_{z=z_k} f(z)$ and $\int_0^\infty f(x)dx = \pi i\sum_{k=1}^{n}\text{Res}_{z=z_k} f(z)$.

26. **Jordan's lemma**

- Suppose that a function $f(z)$ is analytic at all points in the upper half plane $y \ge 0$ that are exterior to a circle $|z| = R_0$, C_R denotes a semicircle $z = Re^{i\theta}, \theta \in [0,\pi]$ where $R > R_0$, and for all points z on C_R there is a positive constant M_R such that $|f(z)| \le M_R$ and $\lim_{R\to\infty} M_R = 0$. **Statement:** Then for every positive constant a, $\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}dz = 0$.
- Alternative statement: If $a > 0$ and P/Q is the quotient of two polynomials such that the degree of Q is greater or equal to the degree of P plus one, then $\lim_{R\to\infty}\int_{C_R} e^{iaz}\frac{P(z)}{Q(z)}dz = 0$ where C_R is the upper half-circle or radius R. If $a < 0$, then use the lower half-plane.

27. **Indented contours**

- Half-poles: Suppose that $f(z)$ has a simple pole at a point $z = x_0$ on the real axis, with a Laurent series representation in the punctured disk $0 < |z - x_0| < R_2$ and with residue B_0, C_ρ denotes the upper clockwise half circle $|z - x_0| = \rho$ where $\rho < R_2$. Then $\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = -\pi i B_0$.
- **Incomplete pole contour** If f has a simple pole at $z = z_0$ and T_r is the circular arc $z = z_0 + re^{i\theta}, \theta_1 \le \theta \le \theta_2$, then $\lim_{r\to 0}\int_{T_r} f(z)dz = i(\theta_2 - \theta_1)\text{Res}(f; z_0)$.

28. **Integrals involving branch points**

- Note that the values of the function across the branch cut can be related by examining the angles just above and below the cut.
- Example: Beta function $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, p > 0, q > 0$. Make the substitution $t = \frac{1}{x+1}$ and use the integral $\int_0^\infty \frac{x^{-a}}{x+1}dx = \frac{\pi}{\sin a\pi}, 0 < a < 1$ to write $B(p, 1-p) = \frac{\pi}{\sin p\pi}$.

- **Function is not even!** If there is no way to relate an integral from 0 to ∞ to an integral over the whole real line, for example $I = \int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)}$, the use the log function. Investigate $\int_0^\infty \frac{Log(z)dz}{(z+1)(z^2+2z+2)}$. Define the branch cut on the positive real axis, then integrate along the branch cut (looping at infinity and around the branch point). Note that the integral above the branch cut has $Log(x)$ in the numerator but the integral below the branch cut has $Log(x) + 2\pi i$ in the numerator. In this way, the integral is performed on the positive real axis only.

29. **Meromorphic functions**
    - A function f is meromorphic in a domain D if it is analytic throughout D except for poles.

30. **Argument principle and Winding Number**
    - Let C be a positively-oriented simple closed contour, and suppose that a function $f(z)$ is meromorphic on the domain interior to C, $f(z)$ is analytic and non-zero on C and counting multiplicities (nth order is counted n times), Z is the number of zeros and P is the number of poles of $f(z)$ inside C. Let $w = f(z)$ be the image Γ of curve C under f. Then Γ traces a closed contour that does not pass through the origin in the w-plane. Let $w_0 = f(z_0)$ with argument ϕ_0. Let ϕ_1 be the argument of w when it returns to w_0. The change in argument $\Delta_C \arg f(z) = \phi_1 - \phi_0$ will be an integral multiple of 2π, and call $\frac{1}{2\pi}\Delta_C \arg f(z)$ the **winding number** of Γ with respect to the origin $w = 0$. **Statement:** Then $\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P$.
    - **Rouche's Theorem:** Let C be a simple closed contour, and suppose that $f(z)$ and $g(z)$ are analytic inside and on C. Let $|f(z)| > |g(z)|$ for all points on C. Then $f(z)$ and $f(z) + g(z)$ will have the same number of zeros, counting multiplicities, inside C. The orientation of C does not matter.
    - **Argument Principle: Short Version:** If f is analytic and non-zero at each point of a simple closed positively oriented contour C and is meromorphic inside C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f)$, where the first term is the number of zeros and the second term is the number of poles of f inside C including multiplicity.

31. **Analytic continuation** Consider two domains D_1 and D_2 with non-empty intersection $D_1 \cap D_2$. If f_1 is analytic in D_1 and $f_2 = f_1$ for each $z \in D_1 \cap D_2$ then f_2 is the unique analytic continuation of f_1 into the domain D_2.

32. **Monodromy theorem**
    - Let $f(z)$ be analytic in domain D and suppose that γ and γ' are two directed smooth curves connecting the point z_1 in D to some point z^*. Suppose that there is some domain D' such that the loop $\Gamma = \{\gamma, -\gamma'\}$ lies in D' and can be continuously deformed to a point in D' and $f(z)$ can be analytically continued along any smooth curve in D'. Then the value at z^* of the analytic continuation of f along γ agrees with the value of its continuation along γ'.

33. **Mappings**
    - The linear transformation $w = Az + B$ is an expansion by $|A|$, rotation by $\arg A$, then a translation by B.
    - The transformation $w = 1/z$ is an inversion with respect to the unit circle composed with a reflection in the real axis (complex conjugate). $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.
      - $w = 1/z$ transforms circles and lines into circles and lines. Consider the arbitrary circle or line $A(x^2 + y^2) + Bx + Cy + D = 0$, where $A \neq 0$ for a circle and $A = 0$ for a line. This can be rewritten as:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2$$

      - Writing $w = u + iv$, we have that $u = \frac{x}{x^2+y^2}, v = \frac{-y}{x^2+y^2}, x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$, and hence the general equation must satisfy:

$$D(u^2 + v^2) + Bu - Cv + A = 0$$

      - A circle ($A \neq 0$) not passing through the origin ($D \neq 0$) in the z-plane is transformed into a circle not passing through the origin in the w-plane.
      - A circle ($A \neq 0$) through the origin ($D = 0$) in the z-plane is transformed into a line that does not pass through the origin in the w-plane.
      - A line ($A = 0$) not passing through the origin ($D \neq 0$) in the z plane is transformed into a circle through the origin in the w-plane.
      - A line ($A = 0$) passing through the origin ($D = 0$) in the z-plane is transformed into a line through the origin in the w-plane.

- Linear fractional transformations: The transformation $T(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$ is a Mobius transformation. The restriction ensures that T is not a constant transformation. We can write it as $w = T(z), Azw + Bz + Cw + D = 0, AD - BC \neq 0$ and hence any equation of the latter type can be put into the linear fractional transformation form. Enlarge the domain of the transformation to the extended z plane by defining $T(\infty) = \frac{a}{c}$ and $T(\frac{-d}{c}) = \infty$ if $c \neq 0$. Then the linear fractional transformation is a one-to-one mapping of the extended z plane onto the extended w plane.
- Inverse linear fractional transformation. Define $T^{-1}(w) = \frac{-dw+b}{cw-a} = z, ad - bc \neq 0$ and on the extended w-plane to be $T^{-1}(\infty) = \frac{-d}{c}, T^{-1}(\frac{a}{c}) = \infty$ if $c \neq 0$.
- $T^{-1}(w) = z \iff T(z) = w$.
- Implicit form for linear fractional transformation. Consider a linear fractional transformation that maps distinct points z_1, z_2, z_3 to w_1, w_2, w_3. Then:

  $$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

  If any of the numbers are infinity, replace that number with its reciprocal and take limits as it goes to zero. This implicit form works with three distinct sets of points (z, w) because three distinct non-linear points uniquely defined a circle, and three collinear points clearly define a line.
- **Cross Ratio** The cross ratio of the 4 points (z, z_1, z_2, z_3) is:

  $$(z, z_1, z_2, z_3) \equiv \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

  and hence the implicit form can be written as $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$.
- The composition of two Mobius transformations is also a Mobius transformation.
- Rotation about a point: $w = e^{i\theta} z + (1 - e^{i\theta}) z_0$ is a mapping that rotates a domain about an angle θ about the point z_0.
- Linear Fractional Transformation of the upper half plane: A transformation of the form $w = e^{i\alpha} \left(\frac{z - z_0}{z - \bar{z}_0} \right), \Im(z_0) > 0$ where $\alpha \in \mathbb{R}$ maps the upper half plane $\Im(z) > 0$ onto the open disk $|w| < 1$ and the boundary of the half plane $\Im(z) = 0$ is mapped onto the boundary $|w| = 1$ of the disk. Implication works in reverse too.
- Mappings by $\sin z$ of horizontal lines: Write $w = \sin z = (\sin x \cosh y) + i(\cos x \sinh y) = u + iv$. Then the vertical $x = c_1$ is transformed into the right-hand branch of the hyperbola $\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1$ using the identity $\cosh^2 y - \sinh^2 y = 1$. This hyperbola has foci at $w = \pm 1$.
- Mappings by $\sin z$ of vertical lines: Given the horizontal line $y = c_2 > 0$ for $-\pi \leq x \leq \pi$, we have the ellipse $\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1$ with foci at $w = \pm 1$.
- Mappings of cosine: Note that $\sin(z + \pi/2) = \cos z$. Hence first translate by $\pi/2$ to the right, then apply the sine transformation.
- Mappings of integer roots: Write $z^{1/n} = \sqrt[n]{r} \exp \frac{i(\theta + 2\pi k)}{n}$, where $k = 0, 1, 2, \ldots, n - 1$ indicates the branch. Let the transformation be applied to the domain $r > 0, -\pi < \theta < \pi$. The image of the mapping is the domain $w = \rho e^{i\phi}, \rho > 0, \frac{(2k-1)\pi}{n} < \phi < \frac{(2k+1)\pi}{n}$.
- **Mapping of unit disk onto itself** The only analytic mappings of the unit disk onto itself are of the form $f(z) = e^{i\theta} \frac{z - \alpha}{\bar{\alpha} z - 1}, |\alpha| < 1$.

34. **Conformal Mapping**

- Let an image Γ of curve C under transformation $f(z)$ be parametrized by $w = f[z(t)], a \leq t \leq b$. Let f be analytic at $z(t_0)$. Then $\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0)$. Call $\theta_0 = \arg z'(t_0)$ the argument of the tangent vector of C at $z(t_0)$ and $\phi_0 = \arg w'(t_0)$, the tangent vector of Γ at $f[z(t_0)]$, $\psi_0 = \arg f'[z(t_0)]$. Then $\phi_0 = \psi_0 + \theta_0$. Call $\psi_0 = \arg f'[z(t_0)] = \phi_0 - \theta_0$ the **angle of rotation**.
- A mapping is **conformal** at a point z_0 if f is analytic there and $f'(z_0) \neq 0$. Note that this implies that the derivative exists and is continuous in a neighbourhood, and that there is a neighbourhood of z_0 throughout which $f'(z) \neq 0$. Geometrically, if we have two curves through a point $z(t_0)$ with tangent vectors θ_1 and θ_2, and the image curves have tangent vectors ϕ_1 and ϕ_2, then we have that $\phi_2 - \phi_1 = \theta_2 - \theta_1$ (both the magnitude and sense are preserved).
- Define the angle between vectors $\vec{v}_1$ and $\vec{v}_2$ to be the angle through which $\vec{v}_1$ must be rotated counterclockwise in order to lie along $\vec{v}_2$.

- A transformation which is conformal at each point in domain D is a conformal mapping. A mapping is conformal in D if f is analytic in D and its derivative f' has no zeros there.
- A mapping that preserves the magnitude of the angle between two smooth arcs but not necessarily the sense is called an **isogonal mapping**.
- **Symmetric with respect to circle** Two points z_1 and z_2 are symmetric with respect to a circle if every straight line or circle passing through z_1 and z_2 intersects the circle orthogonally.
- **Symmetry Principle** Let C_z be a line or circle in the z-plane, and let $w = f(z)$ be any Mobius transformation. Then two points z_1 and z_2 are symmetric with respect to C_z iff they images $w_1 = f(z_1), w_2 = f(z_2)$ are symmetric with respect to the image of C_z under f.
- **Finding symmetric point** Given a circle C with centre a and radius R, the point symmetric to a given point α is

$$\alpha^* = \frac{R^2}{\bar{\alpha} - \bar{a}} + a$$

35. Angle preservation

36. **Local scaling**

    - if z_0 is a critical point of a transformation $w = f(z)$ then there is an integer $m \geq 2$ such that the angle between any two smooth arcs passing through z_0 is multiplied by m under that transformation. m is the smallest positive integer such that $f^{(m)}(z_0) \neq 0$.
    - Scale factor: $|f'(z_0)|$.

37. **Critical points**

    - Suppose f is not a constant function and is analytic at a point z_0. If $f'(z_0) = 0$ then z_0 is a critical point of the transformation $w = f(z)$.

38. **Open mapping property**

    - A function is an open mapping if the image of every open set in its domain is itself open.
    - If f is nonconstant and analytic in a domain D, then its range $f(D) : \{w : w = f(z), z \in D\}$ is an open set.
    - **Riemann Mapping Theorem** Let D be any simply connected domain in the plane other than the entire plane itself. Then there is a one-to-one analytic function that maps D onto the open unit disk. Moreover, one can prescribe an arbitrary point of D and a direction through that point which are to be mapped to the origin and the direction of the positively real axis respectively. Under such restrictions the mapping is unique.

39. **Inverse mappings**

    - **Local one-to-one** If f is analytic at z_0 and $f'(z_0) \neq 0$, then there is an open disk D centered at z_0 such that f is one-to-one on D. It suffices to show that $|f(z_1) - f(z_2)| \geq \left|\frac{f'(z_0)}{2}\right| |z_2 - z_1|$.
    - A transformation $w = f(z)$ that is conformal at a point z_0 has a local inverse there. That is, if $w_0 = f(z_0)$ then there exists a unique transformation $z = g(w)$, defined and analytic in the neighbourhood N of w_0 such that $g(w_0) = z_0$ and $f[g(w)] = w$ for all points w in N. The derivative of $g(w)$ is: $g'(w) = \frac{1}{f'(z)}$.
    - The inverse transformation $z = g(w)$ is itself conformal at w_0.
    - Write $f(z) = u(x, y) + iv(x, y)$ at a point z_0 where f is analytic. The determinant det $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ is the Jacobian of the transformation, and can be written as $J = (u_x)^2 + (v_x)^2 = |f'(z)|^2$ by the CR equations.
    - **Sufficient conditions for the existence of the local inverse:** Suppose $f(z) = u(x, y) + iv(x, y)$ is such that u and v and their first order partial derivatives are continuous at z_0. Suppose that the Jacobian at z_0 is non-zero. Then a unique continuous local inverse $x = x(u, v), y = y(u, v)$ exists on a neighbourhood N of $(u_0, v_0) = (u(x_0, y_0), v(x_0, y_0))$ and maps that point to (x_0, y_0).
    - The components of the local inverse have continuous first-order partial derivatives satisfying:

$$x_u = \frac{1}{J} v_y$$

$$x_v = \frac{-1}{J} u_y$$

$$y_u = \frac{-1}{J} v_x$$

$$y_v = \frac{1}{J} u_x$$

40. **Solving the Laplace equation by conformal mapping of harmonic functions**

- Suppose that an analytic function $w = f(z) = u(x,y) + iv(x,y)$ maps a domain D_z in the z-plane onto a domain D_w in the w-plane. Let $h(u,v)$ be a harmonic function defined on D_w. Then $H(x,y) = h[u(x,y), v(x,y)]$ is harmonic in D_z.
- **Transformation of boundary conditions** Suppose that a transformation $w = f(z) = u(x,y) + iv(x,y)$ is conformal at each point of a smooth arc C and Γ is the image of C under f. Let $h(u,v)$ be a function that satisfies either: $h = h_0 \in \mathbb{R}$ or $\frac{dh}{dn} = 0$ (directional derivative of h normal to Γ) for all points on Γ. **Statement**: Then $H(x,y) = h[u(x,y) + iv(x,y)]$ satisfies the corresponding condition $H = h_0$ or $\frac{dH}{dN} = 0$ at all points on C.
- Under a conformal transformation, the ratio of a directional derivative of H along a smooth arc C in the z-plane to the directional derivative of h along image curve Γ at the corresponding point in the w-plane is $|f'(z)|$.
- Common Harmonic Functions with Boundary Conditions:
  - Washer: $A\text{Log}|z - z_0| + B$ is a washer centered at z_0 that can be fitted to two boundaries at two radii.
  - Wedge: $A\text{Arg}(z - z_0) + B$ is a wedge that can be fitted to two rays passing through z_0. Note that this works for the left/right side of the half-plane. If more conditions are required along the ray, add more wedges.
  - Wall: Basically a wedge made with two antiparallel rays.

41. **Inverse Laplace Transforms**

- Let a complex function $F(s)$ be analytic throughout the finite s-plane except for a finite number of isolated singularities. Let L_R denote a vertical line segment from $\gamma - iR$ to $\gamma + iR$ such that γ is large enough to ensure that the singularities of F lie to the left of that segment. Define $f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{L_R} e^{st} F(s) ds = \frac{1}{2\pi i} P \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds, t > 0$ provided the limit exists. Then $f(t)$ is the inverse Laplace transform of $F(s)$.
- Compute the ILT by using the residue theorem to write $\int_{L_R} e^{st} F(s) ds = 2\pi i \sum_{n=1}^{N} \text{Res}_{s=s_n}[e^{st} F(s)] - \int_{C_R} e^{st} F(s) ds$ where C_R is the semicircle to the left of the segment of radius R centered at $z = \gamma$ that is large enough to enclose all the isolated singularities. Then if $\lim_{R\to\infty} \int_{C_R} e^{st} F(s) ds = 0$, then we have that $f(t) = \sum_{n=1}^{N} \text{Res}_{s=s_n}[e^{st} F(s)], t > 0$.

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