# Ma1c Notes (Apostol)

## 1.1 Definitions

1. **Open n-ball** Let **a** be a given point in $\mathbb{R}^n$ and r be a given positive number. Then the set of all points $x \in \mathbb{R}^n$ such that $||\mathbf{x} - \mathbf{a}|| < r$ is called an open n-ball of radius r and center **a**. Call this $B(\mathbf{a}; r)$.

2. **Interior Point** Let S be a subset of $\mathbb{R}^n$ and assume that $\mathbf{a} \in S$. Then **a** is an interior point of S if there is an open n-ball with center at **a**, all of whose points belong to S. That is, points in the neighbourhood of **a** all belong to S. The interior of a set is the largest open set contained in the set.

3. **Open Set** A set in $\mathbb{R}^n$ is called open if all its points are interior points. That is, S is open iff $S = intS$.

4. **Open Covering (R)** An open covering of a set A in $\mathbb{R}^n$ is a collection $U = \{V_\alpha\}$ of open sets in $\mathbb{R}^n$ such that $A \subseteq \cup_\alpha V_\alpha$. U may be infinite, possibly uncountable.

5. **Subcovering (R)** A subcovering of an opencovering $U = \{V_\alpha\}$ of a set A is a subcollection $U'$ of $U$ such that any point of A belongs to some set in $U'$.

6. **Compact (R)** A set A in $\mathbb{R}^n$ is compact iff any open covering $U = \{V_\alpha\}$ of A contains a finite subcovering. This is a generalization of the notion of being closed and bounded. A closed interval $[a, b] \in \mathbb{R}$ is compact for any real numbers a,b by the Heine-Borel Theorem.

7. **Bounded** A set A in $\mathbb{R}^n$ is bounded if we can enclose it in a closed rectangular box.

8. **Cartesian Product** The Cartesian product of two intervals in $\mathbb{R}^1$ is the set in $\mathbb{R}^2$ defined by $A_1 \times A_2 = \{(a_1, a_2) | a_1 \in A_1 \text{ and } a_2 \in A_2\}$.

9. **Exterior** A point **x** is exterior to a set S if there is an n-ball $B(\mathbf{x})$ containing no points of S. The set of all points in $\mathbb{R}^n$ exterior to S is called the exterior of S, or ext S.

10. **Boundary** A point which is neither exterior to S nor an interior point of S is called a boundary point of S. The set of all boundary points of S is called the boundary of S, or $\partial S$.

11. **Polynomial in n variables** A scalar field P defined on $\mathbb{R}^n$ by a formula of the form $P(x) = \sum_{k_1=0}^{p_1} \cdots \sum_{k_n=0}^{p_n} c_{k_1,\cdots,k_n} x_1^{k_1} \cdots x_n^{k_n}$ is called a polynomial in n variables $x_1, \ldots, x_n$.

12. **Derivative of scalar field with respect to a vector** Given a scalar field $f : S \to R$, where $S \subseteq \mathbb{R}^n$, let **a** be an interior point of S and let **y** be an arbitrary point in $\mathbb{R}^n$. Then the derivative of f at **a** with respect to **y** is denoted by the symbol $f'(\mathbf{a}; \mathbf{y}) = \lim_{h \to 0} \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h}$, when the limit exists.

13. **Directional derivative** If **y** is a unit vector, $f'(\mathbf{a}; \mathbf{y})$ is called the directional derivative of f at **a** in the direction of **y**. Intuitively, it is the instantaneous rate of change of a function, moving through a point with a velocity.

14. **Partial derivative** If $\mathbf{y} = e_k$, the kth unit coordinate vector, the directional derivative $f'(\mathbf{a}; e_k)$ is called the partial derivative with respect to $e_k$, and denoted by $D_k f(\mathbf{a})$. Hence $D_k f(\mathbf{a}) = f'(\mathbf{a}; e_k)$.

15. **Existence of Directional Derivatives through a point does not imply continuity** Also, the existence of all partial derivatives does not imply that the function is differentiable at that point.

16. **Differentiable Scalar Field** Let $f : S \to \mathbb{R}$ be a scalar field defined on a set S in $\mathbb{R}^n$. Let **a** be an interior point of S, and let $B(\mathbf{a}; r)$ be an n-ball lying in S. Let **v** be a vector with $||\mathbf{v}|| < r$ such that $\mathbf{a}+\mathbf{v} \in B(\mathbf{a}; r)$. f is differentiable at **a** if there exists a linear transformation $T_a : \mathbb{R}^n \to \mathbb{R}$ and a scalar function $E(\mathbf{a}, \mathbf{v})$ such that $f(\mathbf{a}+\mathbf{v}) = f(\mathbf{a}) + T_a(\mathbf{v}) + ||\mathbf{v}|| E(\mathbf{a}, \mathbf{v})$ for $||\mathbf{v}|| < r$ where $E(\mathbf{a}, \mathbf{v}) \to 0$ as $||\mathbf{v}|| \to 0$. The linear transformation $T_a$ is called the total derivative of f at **a**.

17. **Differentiable (R)** Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $a \in Int(D)$. Then f is differentiable at a iff there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\lim_{u \to 0} \frac{||f(a+u) - f(a) - L(u)||}{||u||} = 0$

18. **Gradient** $\nabla f(\mathbf{a})$ is the vector whose components are the partial derivatives of f at **a**: $\nabla f(\mathbf{a}) = (D_1 f(\mathbf{a}), \ldots, D_n f(\mathbf{a}))$.

19. **Parametrized curve (R Week 3)** A parametrized curve in $\mathbb{R}^n$ is a continuous function $\alpha : [r, s] \to \mathbb{R}^n$.

20. **Curve (R Week 3)** A curve C in $\mathbb{R}^n$ is the image $C = \alpha([r, s])$ of the map. This curve has an orientation.

21. **Directional Derivative Along a Curve** Let T be the unit tangent vector (norm 1) along the curve. Let **r** describe a curve C, parametrized by t. Then $\nabla f[\mathbf{r}(t)] \cdot T(t)$ is the directional derivative of f along the curve C. This can be written as $\nabla f \cdot T$, or $\frac{df}{ds}$.

22. **Tangent vector (R Week 3)** Let $C$ be a differentiable curve in $\mathbb{R}^n$ parametrized by an $\alpha : [r,s] \to \mathbb{R}^n$. Let $a = \alpha(t_0)$ with $t_0 \in [r,s]$. Then $\alpha'(t_0)$ is called the tangent vector to C at a (in the positive direction). If $\alpha'(t_0) \neq 0$, the tangent space at a is the one-dimensional subspace of $\mathbb{R}^n$ spanned by $\alpha'(t_0)$. If $\alpha'(t_0) = 0$ the tangent space at a is undefined.

23. **Level Set** Let f be a scalar field defined on a set S on $\mathbb{R}^n$. The level set is $L(c) = \{\mathbf{x} | \mathbf{x} \in S, f(\mathbf{x}) = c\}$.

24. **Tangent Plane** A plane through a point $\mathbf{a}$ with normal vector N consists of all points $\mathbf{x}$ satisfying $N \cdot (\mathbf{x} - \mathbf{a}) = 0$.

25. **Tangent Space (R Week 3)** Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable scalar field and $a \in L_c(f)$. If $\nabla f(a) \neq 0$ then we define the tangent space $\Theta_a(L_c(f))$ to $L_c(f)$ at $a$ to be the vector space $\Theta_a(L_c(f)) = \{x \in \mathbb{R}^n | \nabla f(a) \cdot x = 0\}$.

26. **Normal Vector (R Week 3)** A normal vector to $L_c(f)$ at $a$ is a vector $v \in \mathbb{R}^n$ orthogonal to all vectors in $\Theta_a(L_c(f))$.

27. **Multivariable Tangent Plane** The tangent plane to a level surface $L(c)$ at a point $\mathbf{a}$ consists of all $\mathbf{x}$ in $\mathbb{R}^n$ satisfying $\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0$. In three dimensions, write $\nabla f = D_1 f \hat{i} + D_2 f \hat{j} + D_3 f \hat{k}$. Hence we require $D_1 f(\mathbf{a})(x - x_1) + D_2 f(\mathbf{a})(y - y_1) + D_3 f(\mathbf{a})(z - z_1) = 0$.

28. **Differentiably parametrized (R Week 3)** We say that $S \subset \mathbb{R}^n$ can be differentiably parametrized around $a \in S$ if there is a bijective differentiable function $\alpha : \mathbb{R}^k \to S \subset \mathbb{R}^n$ with $\alpha(0) = a$ and so that the linear map $T_0\alpha$ has largest possible rank, namely k. The tangent space to S at $a$ is simply the image of $T_0\alpha$, a linear subspace of $\mathbb{R}^n$. We must have $k \leq n$.

29. **Level set of vector field is an intersection (R Week 3)** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector field with components $(f_1, \ldots, f_m)$. Then for $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ the level set $L_c(f) = \{x \in \mathbb{R}^n | f(x) = c\} = L_{c_1}(f_1) \cap \ldots \cap L_{c_m}(f_m)$, and the tangent space $\Theta_a(L_c(f)) = \{x \in \mathbb{R}^n | (T_\alpha f)(x) = 0\}$ is defined if $T_a f$ has largest possible rank of m. Hence we require that $\nabla f_1(a) \ldots \nabla f_m(a)$ be linearly independent. We also have $\Theta(L_c(f)) = \Theta_a(L_{c_1}(f_1)) \cap \ldots \cap \Theta_a(L_{c_m}(f_m))$ and $\dim_{\mathbb{R}} \Theta_a(L_c(f)) = n - m$.

30. **Derivative of vector field with respect to a vector** Given a vector field $\mathbf{f} : S \to \mathbb{R}^m$ defined on a subset S of $\mathbb{R}^n$. If $\mathbf{a}$ is an interior point of S and if $\mathbf{y}$ is any vector in $\mathbb{R}^n$, we can define the derivative $\mathbf{f}'(\mathbf{a}; \mathbf{y}) = \lim_{h \to 0} \frac{\mathbf{f}(\mathbf{a} + h\mathbf{y}) - \mathbf{f}(\mathbf{a})}{h}$ whenever the limit exists. The derivative is a vector in $\mathbb{R}^n$. Hence we can write this as $\mathbf{f}'(\mathbf{a}; \mathbf{y}) = \sum_{k=1}^m f_k'(\mathbf{a}; \mathbf{y})\mathbf{e}_k$.

31. **Differentiable Vector Field** A vector field $\mathbf{f}$ is differentiable at an interior point $\mathbf{a}$ if there is a linear transformation $\mathbf{T}_a : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathbf{f}(\mathbf{a} + \mathbf{v}) = \mathbf{f}(\mathbf{a}) + \mathbf{T}_a(\mathbf{a}) + ||\mathbf{v}||\mathbf{E}(\mathbf{a}, \mathbf{v})$ where $\mathbf{E}(\mathbf{a}, \mathbf{v}) \to 0$ as $\mathbf{v} \to 0$. The linear transformation $\mathbf{T}_a$ is called the total derivative of $\mathbf{f}$ at $\mathbf{a}$.

32. **Implicit Representation** A surface in 3-space can be described by Cartesian equations of the implicit representation form $F(x, y, z) = 0$.

33. **Jacobian Determinants** The determinant of the Jacobian matrix is given by $\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$.

34. **Relative maximum** The scalar field f is said to have a relative maximum at $\mathbf{a}$ of a set S in $\mathbb{R}^n$ if $f(\mathbf{x}) \leq f(\mathbf{a})$ in some n-ball $B(\mathbf{a})$ lying in S. This is an absolute maximum if the inequality holds for all $x \in S$.

35. **Extremum** A number which is either a relative maximum or a relative minimum of f is called an extremum of f.

36. **Stationary Points** Assume f is differentiable at $\mathbf{a}$. If $\nabla f(\mathbf{a}) = 0$ the point $\mathbf{a}$ is called a stationary point of f.

37. **Saddle Point** A stationary point is called a saddle point if every n-ball $B(\mathbf{a})$ contains points $\mathbf{x}$ such that $f(\mathbf{x}) < f(\mathbf{a})$ and other points such that $f(\mathbf{x}) > f(\mathbf{a})$.

38. **Hessian Matrix** The $n \times n$ matrix of second-order derivatives $D_{ij} f(\mathbf{x})$ is called the Hessian matrix, and is denoted by $H(\mathbf{x}) = [D_{ij} f(\mathbf{x})]_{i,j=1}^n$ whenever the derivatives exist.

39. **n-dimensional interval** An n-dimensional interval is the Cartesian product of n one-dimensional closed intervals. If $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, we write $[\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n] = \{(x_1, \ldots, x_n) | x_1 \in [a_1, b_1], \ldots, x_n \in [a_n, b_n]\}$.

40. **Span** THe span of a scalar field f on n-dimensional interval $[\mathbf{a}, \mathbf{b}]$ is the difference between the maximum and minimum values of f on the interval.

41. **Partition** The n-dimensional partition of the interval $[\mathbf{a}, \mathbf{b}]$ is the Cartesian product $P = P_1 \times \cdots \times P_n$, where each $P_k = \{x_0, x_1, \ldots, x_{r-1}, x_r\}$ such that $a_k = x_0 \leq x_1 \leq \cdots \leq x_{r-1} \leq x_r = b_k$.

42. **Partition (R Week 4)** A partition of R is a finite collection $P$ of subrectangular closed boxes $S_1, \ldots, S_r \subseteq R$ such that (i) $R = \cup_{j=1}^{r} S_j$ and (ii) the interiors of $S_i$ and $S_j$ have no intersection for all $i \neq j$.

43. **Refinement (R Week 4)** A refinement of a partition $P = \{S_j\}_{j=1}^{r}$ of $R$ is another partition $P' = \{S'_k\}_{k=1}^{m}$ with each $S'_k$ contained in some $S_j$.

44. **Smooth vs Piecewise** Let $J = [a, b]$ be a finite closed interval in $\mathbb{R}^1$. A function $\mathbf{a} : J \to \mathbb{R}^n$ which is continuous on J is called a continuous path in n-space. The path is smooth if the derivative $\mathbf{a}'$ exists and is continuous on the open interval $(a, b)$. The path is piecewise if the interval $[a, b]$ can be partitioned into a finite number of subintervals in each of which the path is smooth.

45. **Line Integral** Let $\mathbf{a}$ be a piecewise smooth path in n-space defined on an interval $[a, b]$, and let $\mathbf{f}$ be a vector field defined and bounded on the graph of $\mathbf{a}$. The line integral of $\mathbf{f}$ along $\mathbf{a}$ is denoted by the symbol $\int \mathbf{f} \cdot d\mathbf{a}$ and is defined by the equation $\int \mathbf{f} \cdot d\mathbf{a} = \int_a^b \mathbf{f}[\mathbf{a}(t)] \cdot \mathbf{a}'(t) dt$, whenever the integral exists. In terms of components, this is $\sum_{k=1}^{n} \int_a^b f_k[\mathbf{a}(t)] \alpha'_k(t) dt = \int f_1 d\alpha_1 + \cdots + f_n d\alpha_n$.

46. **Orientation** Let $\mathbf{a}$ be a continuous path defined on an interval $[a, b]$, and let u be a real-valued function that is differnetiable, with $u'$ never zero on an interval $[c, d]$, and such that the range of u is $[a, b]$. Then the function $\mathbf{b}$ defined on $[c, d]$ by the equation $\mathbf{b}(t) = \mathbf{a}[u(t)]$ is a continuous path having the same graph as $\mathbf{a}$. Two paths $\mathbf{a}$ and $\mathbf{b}$ so related are called equivalent. If the derivative of u is always positive on $[c, d]$, the function u is increasing and the two paths $\mathbf{a}$ and $\mathbf{b}$ trace out the curve C in the same direction (u is orientation-preserving). If the derivative of u is always negative, $\mathbf{a}$ and $\mathbf{b}$ trace out C in opposite directions (u is orientation-reversing).

47. **Rectifiable Curve** A rectifiable curve is a curve with finite length.

48. **Arc-length function** Let $\mathbf{a}$ be a path with $\mathbf{a}'$ continuous on an interval $[a, b]$. The arc-length function is $s(t) = \int_a^t ||\mathbf{a}'(u)|| du$. The derivative is $s'(t) = ||\mathbf{a}'(t)||$.

49. **Line integral with respect to arc length** Let $\mathbf{a}$ be a path with $\mathbf{a}'$ continuous on an interval $[a, b]$. Let $\phi$ be a scalar field defined and bounded on C, the graph of $\mathbf{a}$. The line integral of $\phi$ with respect to arc length along C is denoted by $\int_C \phi ds = \int_a^b \phi[\mathbf{a}(t)] s'(t) dt$ whenver the integral exists. If $\phi$ is obtained by the dot product of a vector field $\mathbf{f}$ defined on C and the unit tangent vector $\mathbf{T}(t) = \frac{d\mathbf{a}}{ds}$, then $\int_C \phi ds = \int_C \mathbf{f} \cdot d\mathbf{a}$.

50. **Flow Integral** When $\mathbf{f}$ denotes a velocity field and $\mathbf{T}$(t) is the unit tangent vector, then the line integral $\int_C \mathbf{f} \cdot \mathbf{T} ds$ is the flow integral of $\mathbf{f}$ along C. When C is closed, the flow integral is called the circulation of $\mathbf{f}$ along C.

51. **Connected Set** Let S be an open set in $\mathbf{R}^n$. The set S is connected if every pair of points in S can be joined by a piecewise smooth path whose graph lies in S.

52. **Disconnected Set** An open set S is said to be disconnected if S is the union of two or more disjoint non-empty open sets.

53. **Convex Set** A set S in $\mathbb{R}^n$ is called convex if every pair of points in S can be joined by a line segment, all of whose points lie in S. Every open convex set is connected.

54. **Exact Differential Equation** A differential equation $P(x, y)dx + Q(x, y)dy = 0$ is called exact in S if there is an associated vector field $\mathbf{V}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ such that $\mathbf{V}(x, y) = \nabla\phi(x, y)$ is the gradient of a scalar potential for each point in S.

55. **Step Function** A function f defined on a rectangle Q is said to be a step function if a partition P of Q exists such that f is constant on each of the open subrectangles of P.

56. **Double Integral of a Step Function** Let f be a step function which takes the constant value $c_{ij}$ on the open subrectangle $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$ of a rectangle Q. The double integral of f over Q is defined by the formula $\iint_Q f = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \cdot (x_i - x_{i-1})(y_j - y_{j-1})$.

57. **Integral of a bounded function** If there is one and only one number I such that $\iint_Q s \leq I \leq \iint_Q t$ for every pair of step functions satisfying $s(x, y) \leq f(x, y) \leq t(x, y)$, the number I is called the double integral of f over Q. When such an I exists, f is said to be integrable on Q.

58. **Bounded Set of Content Zero** Let A be a bounded subset of the plane. The set A is said to have content zero if for every $\epsilon > 0$, there is a finite set of rectangles whose union contains A and the sum of whose areas does not exceed $\epsilon$. Hence a bounded plane set of content zero can be enclosed in a union of rectangles whose total area is arbitrarily small.

59. **Ordinate Set** Let S be a type I region bounded between $a \leq x \leq b, \phi_1(x) \leq y \phi_2(x)$. If f is non-negative, the set of points $(x, y, z) \in \mathbb{R}^3$ such that $(x, y) \in S$ and $0 \leq z \leq f(x, y)$ is called the ordinate set of f over S.

60. **Closed curves** Suppose a curve C is described by a continuous vector-valued function $\alpha$ defined on an interval $[a, b]$. If $\alpha(a) = \alpha(b)$, the curve is closed.

61. **Simple Closed Curve/Jordan Curve** A closed curve such that $\alpha(t_1) \neq \alpha(t_2)$ for every $t_1 \neq t_2$ in the half open interval $(a, b]$ is a simple closed curve. A simple closed curve that lies in a plane is called a Jordan curve. A Jordan curve decomposes the plane into two disjoint open connected sets having the curve C as their common boundary. One region is bounded, and is called the interior (or inner region) of C. The other is unbounded and is called the exterior (or outer region) of C.

62. **Simply Connected Plane Set** Let S be an open connected set in the plane. Then S is called simply connected if, for every Jordan curve C which lies in S, the inner region of C is also a subset of S.

63. **Winding Number** Let C be a piecewise smooth closed curve in the plane described by a vector-valued function $\alpha$ defined on an interval $[a, b]$, say, $\alpha(t) = (X(t), Y(t)), t \in [a, b]$. Let $P_0 = (x_0, y_0)$ be a point which does not lie on the curve C. Then the winding number of $\alpha$ with respect to the point $P_0$ is denoted by $W(\alpha, P_0)$ and is defined to be the value of $W(\alpha, P_0) \equiv \frac{1}{2\pi} \int_a^b \left[ \frac{(X(t)-x_0)Y'(t)}{r^2} - \frac{(Y(t)-y_0)X'(t)}{r^2} \right] dt$, where $r^2 = (X(t) - x_0)^2 + (Y(t) - y_0)^2$. The value of this integral is always an integer. If C is a Jordan curve, this integer is 0 if $P_0$ is outside C and is +1 if $P_0$ is inside C and $\alpha$ traces out C in a positive direction, and is $-1$ if $P_0$ is inside C and $\alpha$ traces out C in a negative direction.

64. **Parametric Representations** Of a sphere: $x = a \cos u \cos v, y = a \sin u \cos v, z = a \sin v$ with $(u, v) \in [0, 2\pi] \times [-\pi/2, \pi/2]$. Of a cone $x = v \sin \alpha \cos u, y = v \sin \alpha \sin u, z = v \cos \alpha$, $\alpha$ is half the vertex angle, cone points in the $z$ direction, $(u, v) \in [0, 2\pi] \times [0, h]$.

65. **Fundamental Vector Product** Consider a surface described by $r(u, v) = X(u, v)i + Y(u, v)j + Z(u, v)k$, where $(u, v) \in T$. If $X, Y, Z$ are differentiable on T, define the two vectors $\frac{\partial r}{\partial u} = (\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u})$, $\frac{\partial r}{\partial v} = (\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v})$. The fundamental vector product of the representation r is the cross product of the two vectors:

$$\begin{aligned} \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} &= \begin{vmatrix} \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix} i + \begin{vmatrix} \frac{\partial Z}{\partial u} & \frac{\partial X}{\partial u} \\ \frac{\partial Z}{\partial v} & \frac{\partial X}{\partial v} \end{vmatrix} j + \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix} k \\ &= \frac{\partial(Y, Z)}{\partial(u, v)} i + \frac{\partial(Z, X)}{\partial(u, v)} j + \frac{\partial(X, Y)}{\partial(u, v)} k \end{aligned}$$

Themagnitude of the fundamental vector product may be thought of as a local magnification factor for areas.

66. **Regular Point** If point $(u, v) \in T$ is such that $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ are continuous and the fundamental vector product is non-zero, then $r(u, v)$ is a regular point of $r$. At each regular point, the vectors $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ determine a tangent plane having the vector $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ as a normal (section 12.3).

67. **Singular Point** A point $r(u, v)$ at whch $\frac{\partial r}{\partial u}$ or $\frac{\partial r}{\partial v}$ fails to be continuous or $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = 0$ is a singular point of $r$.

68. **Smooth Surface** A surface $r(T)$ is smooth if all its points are regular points.

69. **Area of a Parametric Surface** The area of S is $a(S) = \iint_T \left|\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|\right| dudv$. If $S$ is given explicitly as $z = f(x, y)$, then $\left|\left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right|\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$. If $S$ is defined implicitly, then we have to use the Jacobian form: $\left|\left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right|\right| = \sqrt{\left(\frac{\partial(Y,Z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(Z,X)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(X,Y)}{\partial(u,v)}\right)^2}$.

70. **Surface Integral of Scalar Field (R Week 8)** Let $S = r(T)$ be a parametric surface described by a differentiable function $r$ defined on a region T in the $uv$-plane, and let $f$ be a scalar field defined and bounded on S. The surface integral of $f$ over S is denoted by the symbol $\iint_{r(T)} f dS$ or $\iint_S f(x, y, z) dS$ and is defined by $\iint_{r(T)} f dS = \iint_T f[r(u, v)] \left|\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|\right| dudv$ whenver the double integral on the right exists.

71. **Surface Integral of Vector Field (R Week 8)** Let $F$ be a vector field on $\Phi$. Then the surface integral of $F$ over $\Phi$, denoted $\iint_\Phi F \cdot n dS$, is defined by $\iint_\Phi F \cdot n dS = \iint_T F(\phi(u, v)) \cdot \left(\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right) dudv$, where $n$ is the unit normal vector to $\Phi$ at $\phi(u, v)$. If $F = (P, Q, R)$, write $\iint_\Phi F \cdot n dS = \iint_\Phi P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$. The bilinear vector product can be expanded as $da \wedge db = \frac{\partial(A,B)}{\partial(u,v)} dudv$, where $a, b = x, y, z$.

72. **Smoothly Equivalent (R Week 7a)** Suppose a function $r$ maps a region A in the $uv$-plane onto a parametric surface $r(A)$. Suppose also that $A$ is the image of a region $B$ in the $st$-plane under a one-to-one continuously differentiable mapping $G$ given by $G(s,t) = U(s,t)i + V(s,t)j, (s,t) \in B$. Consider the function $R$ on B by the equation $R(s,t) = r[G(s,t)]$. Two functions $r$ and $R$ so related are called smoothly equivalent, and describe the same surface: $r(A)$ and $R(B)$ are identical as point sets.

73. **Simply Connected (R Week 8)** A connected open set $R \subseteq \mathbb{R}^2$ is called simply connected if for any Jordan curve $C \subset R$, the interior of C lies completely in R. If a region is not simply connected, it is called multiply connected.

74. **Primitive mapping (R Week 8)** Let $D$ be an open set in $\mathbb{R}$. A mapping $\phi : D \to \mathbb{R}^2$ is primitive if it is either of the form $\tilde{g} : (u,v) \mapsto (u, g(u,v))$ or $\tilde{h} : (u,v) \mapsto (h(u,v), v)$ with $g, h$ in $C^1$ and $\partial g/\partial v, \partial/\partial u$ nowhere vanishing on $D$.

75. **Parametrized k-fold (R Week 8)** Let $n, k$ be positive integers with $k \leq n$. A subset $\Phi$ of $\mathbb{R}^n$ is called a parametrized k-fold iff there exists a bounded, connected region T in $\mathbb{R}^k$ together with a $C^1$ injective mapping $\phi : T \to \mathbb{R}^n, u \mapsto (x_1(u), x_2(u), \ldots, x_n(u))$ such that $\phi(T) = \Phi$. When $k = 2$, this is a parametrized surface, and when $k = 1$, it is a parametrized curve.

76. **Orientable surface (R Week 9)** A smooth surface $\Phi$ is orientable if as we move the inward normal along a curve on $\Phi$ and come back to the initial point, then the inward normal continues to remain the inward normal. A smooth closed surface is orientable with two possible orientations (inward/outward).

77. **K-Forms (R Week 9)** A 0-form on $\mathbb{R}^3$ is a scalar valued function $f$, a 1-form is a vector valued function $(P,Q,R)$, which we write $\omega = Pdx + Qdy + Rdz$. A 2-form is a vector valued function $(P,Q,R)$ which we write $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$. A 3-form is a scalar valued function $f$ which we write as $fdx \wedge dy \wedge dz$. The degree of a k-form is k.

78. **Derivative of k-forms (R Week 9)** The derivative of a k-form is a k+1-form determined as follows: If $f$ is a 0-form, then $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. If $\omega = Pdx + Qdy + Rdz$ is a 1-form, then $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz$. If $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ is a 2-form, then $d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz$. If $\omega = fdx \wedge dy \wedge dz$ is a 3-form, then $d\omega = 0$.

## 1.2 Theorems

**Theorem 5 (R Week 1)** Let A be a subset of $\mathbb{R}^n$ which is closed and bounded. Then A is compact.

**Corollary 1 (R Week 1)** Closed balls and spheres in $\mathbb{R}^n$ are compact.

**Proposition 1 (R Week 1)** Let $f : D \to \mathbb{R}^m$ be a vector field. Then f is continuous at every point $a \in D$ iff the following holds: for every open set W of $\mathbb{R}^m$, its inverse image $f^{-1}(W) := \{x \in D | f(x) \in W\} \in D$ is open.

**Proposition 2 (R Week 1)** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then, given any compact set C of $\mathbb{R}^n, f(C)$ is compact.

**Corollary 2 (R Week 1)** Any continuous real valued function f on a compact set $C \subset \mathbb{R}^n$ has a maximum and a minimum, i.e. there are $x_{max}$ and $x_{min} \in C$ so that $f(x_{min}) \leq f(x) \leq f(x_{max})$ for all $x \in C$.

**Theorem 8.1** If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = \mathbf{c}$, then we also have:
(a) $\lim_{\mathbf{x}\to\mathbf{a}}[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})] = \mathbf{b} + \mathbf{c}$ (not proven)
(b) $\lim_{\mathbf{x}\to\mathbf{a}} \lambda\mathbf{f}(\mathbf{x}) = \lambda\mathbf{b}$ (not proven)
(c) $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \mathbf{b} \cdot \mathbf{c}$
(d) $\lim_{\mathbf{x}\to\mathbf{a}} ||\mathbf{f}(\mathbf{x})|| = ||\mathbf{b}||$.

**Continuity** A function $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is continous at a point iff each component $f_k$ is continuous at that point. (Not proven completely in text)

**Theorem 8.2** Let $\mathbf{f}$ and $\mathbf{g}$ be functions such that the composite function $\mathbf{f} \circ \mathbf{g}$ is defined at $\mathbf{a}$, where $(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x}))$. If $\mathbf{g}$ is continuous at $\mathbf{a}$ and $\mathbf{f}$ is continuous at $\mathbf{g}(\mathbf{a})$, then the composition $\mathbf{f} \circ \mathbf{g}$ is continuous at $\mathbf{a}$.

**Theorem 8.3** Let $g(t) = f(\mathbf{a} + t\mathbf{y})$. If one of the derivatives $g(t)$ or $f'(\mathbf{a} + t\mathbf{y}; \mathbf{y})$ exists, then the other also exists and the two are equal. In particular, $g'(0) = f'(\mathbf{a}; \mathbf{y})$.

**Theorem 8.4** Mean value theorem for derivatives of scalar fields: Assume $f'(\mathbf{a}+t\mathbf{y};\mathbf{y})$ exists for each t in the interval $0 \le t \le 1$. Then for some real $\theta$ in the open interval $0 < \theta < 1$ we have $f(\mathbf{a}+\mathbf{y}) - f(\mathbf{a}) = f'(\mathbf{z};\mathbf{y})$, where $\mathbf{z} = \mathbf{a} + \theta\mathbf{y}$.

**Theorem 8.5** Assume scalar field f is differentiable at $\mathbf{a}$ with total derivative $T_a$. Then the derivative $f'(\mathbf{a};\mathbf{y})$ exists for every $\mathbf{y}$ in $\mathbb{R}^n$ and we have $T_a(\mathbf{y}) = f'(\mathbf{a};\mathbf{y})$. $f'(\mathbf{a};\mathbf{y})$ is a linear combination of the components of $\mathbf{y}$. If $\mathbf{y} = (y_1, \ldots, y_n)$, then we have $f'(\mathbf{a};\mathbf{y}) = \sum_{k=1}^{n} D_k f(\mathbf{a}) y_k$. This can be written as $f'(\mathbf{a};\mathbf{y}) = \nabla f(\mathbf{a}) \cdot \mathbf{y}$.

**Lemma 1 (R Week 2)** The total derivative linear map, if it exists, is unique.

**Theorem 8.6** If a scalar field f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

**Theorem 8.7** Sufficient condition for differentiability: Assume that the partial derivatives $D_1 f, \ldots, D_n f$ exist in some n-ball $B(\mathbf{a})$ and are continuous at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$. Call f continuously differentiable.

**Theorem 8.8** Multivariable Chain Rule. Let f be a scalar field defined on an open set S in $\mathbb{R}^n$ and let $\mathbf{r}$ be a vector-valued function which maps an interval J from $\mathbb{R}^1$ into S. Define the composite function $g = f \circ \mathbf{r}$ on J by the equation $g(t) = f[\mathbf{r}(t)] \quad if \quad t \in J$. Let t be a point in J at which $\mathbf{r}'(t)$ exists and assume that f is differentiable at $\mathbf{r}(t)$. Then $g'(t)$ exists and is equal to the dot product $g'(t) = \nabla f(\mathbf{a}) \cdot \mathbf{r}'(t)$, where $\mathbf{a} = \mathbf{r}(t)$.

**Theorem 8.9** Assume vector field $\mathbf{f}$ is differentiable at $\mathbf{a}$ with total derivative $\mathbf{T}_a$. Then the derivative $\mathbf{f}'(\mathbf{a};\mathbf{y})$ exists for every $\mathbf{y}$ (?) in $\mathbb{R}^n$ and $\mathbf{T}_a(\mathbf{y}) = \mathbf{f}'(\mathbf{a};\mathbf{y})$. Let $\mathbf{f} = (f_1, \ldots, f_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Then $\mathbf{T}_a(\mathbf{y}) = \sum_{k=1}^{m} \nabla f_k(\mathbf{a}) \cdot \mathbf{y}\mathbf{e}_k = (\nabla f_1(\mathbf{a}) \cdot \mathbf{y}, \ldots, \nabla f_m(\mathbf{a}) \cdot \mathbf{y})$. This can be written as $\mathbf{T}_a(\mathbf{y}) = D\mathbf{f}(\mathbf{a})\mathbf{y}$, where $D\mathbf{f}(\mathbf{a})$ is the $m \times n$ Jacobian matrix whose $k$th row is $\nabla f_k(\mathbf{a})$. The $i, j$th entry is the partial derivative $D_j f_i(\mathbf{a})$. Hence,

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} D_1 f_1(\mathbf{a}) & D_2 f_1(\mathbf{a}) & \cdots & D_n f_1(\mathbf{a}) \\ D_1 f_2(\mathbf{a}) & D_2 f_2(\mathbf{a}) & \cdots & D_n f_2(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_m(\mathbf{a}) & D_2 f_m(\mathbf{a}) & \cdots & D_n f_m(\mathbf{a}) \end{bmatrix}$$

The Jacobian matrix is defined at each point where the $mn$ partial derivatives $D_j f_i(\mathbf{a})$ exist. The Jacobian matrix can also be written as $\mathbf{f}'(\mathbf{a})$, and is the matrix representation for the linear transformation $\mathbf{T}_a$.

**Linear Map continuous: Lemma 4 (R Week 2)** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $\exists c > 0$ such that $||Tv|| \le c||v||$ for any $v \in \mathbb{R}^n$.

**Theorem 8.10** If a vector field $\mathbf{f}$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$. (R): However, this does not mean that the partial derivatives are continuous at a.

**Lemma 2 (R Week 2)** Let $f_1, \ldots, f_m$ be the component scalar fields of vector field f. Then f is differentiable at a iff each $f_i$ is differentiable at a.

**Theorem 8.11** Vector Field Chain Rule: Let $\mathbf{f}$ and $\mathbf{g}$ be vector fields such that the composition $\mathbf{h} = \mathbf{f} \circ \mathbf{g}$ is defined in a neighbourhood of a point $\mathbf{a}$. Assume that $\mathbf{g}$ is differentiable at $\mathbf{a}$, with total derivative $\mathbf{g}'(\mathbf{a})$. Let $\mathbf{b} = \mathbf{g}(\mathbf{a})$ and assume that $\mathbf{f}$ is differentiable at $\mathbf{b}$, with total derivative $\mathbf{f}'(\mathbf{b})$. Then $\mathbf{h}$ is differentiable at $\mathbf{a}$, and the total derivative $\mathbf{h}'(\mathbf{a})$ is given by $\mathbf{h}'(\mathbf{a}) = \mathbf{f}'(\mathbf{b}) \circ \mathbf{g}'(\mathbf{a})$. Note that this can be written as $\mathbf{h}'(\mathbf{a}) = \mathbf{f}'(\mathbf{g}(\mathbf{a})) \circ \mathbf{g}'(\mathbf{a})$. In Jacobian matrix form, $D\mathbf{h}(\mathbf{a}) = D\mathbf{f}(\mathbf{b}) D\mathbf{g}(\mathbf{a})$.

**Total Derivatives of composite functions (R Theorem 1f)** Assume $T_a f$ and $T_a g$ exist. Then $T_a(f+g)$ exists and $T_a(f+g) = T_a f + T_a g$. If f and g are scalar fields, and are differentiable at a, then $T_a(fg) = f(a) T_a g + g(a) T_a f$ and $T_a(f/g) = \frac{g(a) T_a f - f(a) T_a g}{g(a)^2}$ if $g(a) \neq 0$.

**Theorem 8.12** Sufficient condition for equality of mixed partial derivatives: Assume f is a scalar field such that the partial derivatives $D_1 f$, $D_2 f$, $D_{1,2} f$ and $D_{2,1} f$ exist on an open set S. If $(a, b)$ is a point in S at which both $D_{1,2} f$ and $D_{2,1} f$ are continuous, we have $D_{1,2} f(a, b) = D_{2,1} f(a, b)$.

**Tangent vector in tangent space (R Week 3 Proposition 1)** Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable scalar field, with $\alpha : \mathbb{R} \to \text{Ł}_c(f)$ a curve whcih is differentiable at $t_0 \in \mathbb{R}$ and so that $a = \alpha(t_0)$ is a smooth point of $L_c(f)$. Then $\alpha'(t_0) \in \Theta_a(L_c(f))$.

**Theorem 8.13** A stronger version of Theorem 8.12. Let f be a scalar field such that the partial derivatives $D_1 f$, $D_2 f$ and $D_{2,1} f$ exist on an open set S containing $(a, b)$. Assume further that $D_{2,1} f$ is continuous on S. Then the derivative $D_{1,2} f(a, b)$

exists and we have $D_{1,2}f(a,b) = D_{2,1}f(a,b)$.

**Theorem 9.1** Let g be differentiable on $\mathbb{R}^1$ and let f be the scalar field defined on $\mathbb{R}^2$ by the equation $f(x,y) = g(bx - ay)$, where a and b are constants, not both zero. Then f satisfies the first-order partial differential equation $a\frac{\partial f(x,y)}{\partial x} + b\frac{\partial f(x,y)}{\partial y} = 0$ everywhere in $\mathbb{R}^2$. Also, every differentiable solution of the PDE has the form of f for some g.

**Theorem 9.2** D'Alembert's solution of the wave equation. Let F and G be given functions such that G is differentiable and F is twice differentiable on $\mathbb{R}^1$. Then the function f given by the formula:

$$f(x,t) = \frac{F(x+ct) + F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$

satisfies the one-dimensional wave equation: $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$ and the initial conditions $f(x,0) = F(x)$, $D_2 f(x,0) = G(x)$ (i.e. differentiation with respect to the second variable, t). Conversely, any function f with equal mixed partials which satisfies the initial conditions and the wave equation necessarily has the form above.

**Theorem 9.3** Let F be a scalar field differentiable on an open set T in $\mathbb{R}^n$. Assume that the equation $F(x_1,\ldots,x_n) = 0$ defines $x_n$ implicitly as a differentiable function of $x_1,\ldots,x_{n-1}$, say, $x_n = f(x_1,\ldots,x_{n-1})$ for all points $(x_1,\ldots,x_{n-1})$ in some open set S in $\mathbb{R}^{n-1}$. Then for each $k = 1,2,\ldots,n-1$, the partial derivative $D_k f$ is given by the formula $D_k f = -\frac{D_k F}{D_n F}$ at those points at which $D_n F \neq 0$. The partial derivatives involving $D_k F$ and $D_n F$ are evaluated at the point $(x_1,\ldots,x_{n-1}, f(x_1,\ldots,x_{n-1}))$.

**Theorem 9.4** Second-order Taylor Formula for Scalar Fields: Let f be a scalar field with continuous second-order partial derivatives $D_{ij}f$ in an n-ball $B(\mathbf{a})$ (so that the mixed derivatives are symmetric). Then for all $y \in \mathbb{R}^n$ such that $\mathbf{a}+\mathbf{y} \in B(\mathbf{a})$, we have:

$$f(\mathbf{a}+\mathbf{y}) - f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{y} + \frac{1}{2!}\mathbf{y}H(\mathbf{a}+c\mathbf{y})\mathbf{y}^t, 0 < c < 1$$

This can also be written in the form:

$$f(\mathbf{a}+\mathbf{y}) - f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{y} + \frac{1}{2!}\mathbf{y}H(\mathbf{a})\mathbf{y}^t + ||\mathbf{y}||^2 E_2(\mathbf{a},\mathbf{y})$$

where $E_2(\mathbf{a},\mathbf{y}) \to 0$ as $\mathbf{y} \to 0$.

**Theorem 9.5** Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix, and let $Q(\mathbf{y}) = \mathbf{y}A\mathbf{y}^t = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}y_i y_j$. Then we have (a) $Q(\mathbf{y}) > 0$ for all $\mathbf{y} \neq 0$ iff all the eigenvalues of A are positive (positive definite) (b) $Q(\mathbf{y}) < 0$ for all $\mathbf{y} \neq 0$ iff all the eigenvalues of A are negative (negative definite).

**Theorem 9.6** Let f be a scalar field with continuous second-order partial derivatives $D_{ij}f$ in an n-ball $B(\mathbf{a})$, and let $H(\mathbf{a})$ denote the Hessian matrix at a stationary point $\mathbf{a}$. Then we have (a) If all the eigenvalues of $H(\mathbf{a})$ are positive, f has a relative minimum at $\mathbf{a}$ (b) If all the eigenvalues of $H(\mathbf{a})$ are negative, f has a relative maximum at $\mathbf{a}$ (c) If $H(\mathbf{a})$ has both positive and negative eigenvalues, then f has a saddle point at $\mathbf{a}$. If all the eigenvalues of $H(\mathbf{a})$ are zero, there is no information concerning the stationary point.

**Theorem 9.7** Let $\mathbf{a}$ be a stationary point of a scalar field $f(x_1,x_2)$ with continuous second-order partial derivatives in a 2-ball $B(\mathbf{a})$. Let $A = D_{1,1}f(\mathbf{a})$, $B = D_{1,2}f(\mathbf{a})$, $C = D_{2,2}f(\mathbf{a})$, and let $\Delta = \det H(\mathbf{a}) = \det \begin{pmatrix} A & B \\ B & C \end{pmatrix} = AC - B^2$. Then we have (a) if $\Delta < 0$, f has a saddle point at $\mathbf{a}$, (b) if $\Delta > 0$ and $A > 0$, f has a relative minimum at $\mathbf{a}$, (c) if $\Delta > 0$ and $A < 0$, f has a relative maximum at $\mathbf{a}$, (d) if $\Delta = 0$, the test is inconclusive.

**Method of Lagrange's multipliers** If a scalar field $f(x_1,\ldots,x_n)$ has a relative extremum when it is subject to m constraints, say $g_1(x_1,\ldots,x_n) = 0,\ldots,g_m(x_1,\ldots,x_n) = 0$, where $m < n$, then there exist m scalars $\lambda_1,\ldots,\lambda_m$ such that $\nabla f = \lambda_1\nabla g_1 + \ldots + \lambda_m \nabla g_m$ at each extremum point. Consider the system of $n+m$ equations, and solve these for the $n+m$ unknowns $x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m$.

**Theorem 9.8** Boundedness Theorem for Continuous Scalar Fields. If f is a scalar field continuous at each point of a closed interval $[\mathbf{a},\mathbf{b}] \in \mathbb{R}^n$, then f is bounded on $[\mathbf{a},\mathbf{b}]$. That is, there is a number $C \geq 0$ such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in [\mathbf{a},\mathbf{b}]$.

**Theorem 9.9** Extreme-value Theorem for Continuous Scalar Fields: If f is continuous on a closed interval $[\mathbf{a}, \mathbf{b}] \in \mathbb{R}^n$, then there exist points $\mathbf{c}$ and $\mathbf{d}$ in $[\mathbf{a}, \mathbf{b}]$ such that $f(\mathbf{c}) = \sup f$ and $f(\mathbf{d}) = \inf f$.

**Theorem 9.10** Let f be a scalar field continuous on a closed interval $[\mathbf{a}, \mathbf{b}] \in \mathbb{R}^n$. Then for every $\epsilon > 0$ there is a partition of $[\mathbf{a}, \mathbf{b}]$ into a finite number of subintervals such that the span of f in every subinterval is less than $\epsilon$.

**Theorem 10.1** Let $\mathbf{a}$ and $\mathbf{b}$ be equivalent piecewise smooth paths. Then we have $\int_C \mathbf{f} \cdot d\mathbf{a} = \int_C \mathbf{f} \cdot d\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ trace out C in the same direction, and $\int_C \mathbf{f} \cdot d\mathbf{a} = -\int_C \mathbf{f} \cdot d\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ trace out C in the opposite direction.

**Theorem 10.2** Second Fundamental theorem of calculus for line integrals: Let $\phi$ be a real function that is continuous on a closed interval $[a, b]$ and assume that the integral $\int_a^b \phi'(t)dt$ exists. If $\phi'$ is continuous on the open interval $(a, b)$, we have $\int_a^b \phi'(t)dt = \phi(b) - \phi(a)$.

**Second Fundamental Theorem of Calculus for Line Integrals (R)** Let $g$ be a differentiable scalar field with continuous gradient $\nabla g$ on an open set D in $\mathbb{R}^n$. Then, for any two points $P, Q \in D$ joined by a piecewise $C^1$ path C lying completely in D and parametrized by $\alpha : [a, b] \to D$ with $\alpha(a) = P$ and $\alpha(b) = Q$, we have $\int_C \nabla g \cdot d\alpha = g(Q) - g(P)$.

**Week 6 Corollary 1** Let $g$ be a differentiable scalar field with continuous gradient $\nabla g$ on an open set D in $\mathbb{R}^n$. Then for any point $P \in D$ and any piecewise $C^1$ path connecting P to itself, we have $\int_C \nabla g \cdot d\alpha = 0$.

**Theorem 10.3** Second Fundamental theorem of calculus for line integrals (multivariable): Let $\phi$ be a differentiable scalar field with a continuous gradient $\nabla\phi$ on an open connected set S in $\mathbb{R}^n$. Then for any two points $\mathbf{a}$ and $\mathbf{b}$ joined by a piecewise smooth path $\alpha$ in S we have $\int_a^b \nabla\phi \cdot d\alpha = \phi(\mathbf{b}) - \phi(\mathbf{a})$. Note this is independent of the path in any open connected set whenever the gradient is continuous.

**Theorem 10.4** First Fundamental Theorem for Line Integrals: Let $\mathbf{f}$ be a vector field that is continuous on an open connected set S in $\mathbb{R}^n$, and assume that the line integral of $\mathbf{f}$ is independent of the path in S. Let $\mathbf{a}$ be a fixed point of S and define a scalar field $\phi$ on S by the equation $\phi(\mathbf{x}) = \int_a^x \mathbf{f} \cdot d\alpha$, where $\alpha$ is any piecewise smooth path in S joining $\mathbf{a}$ to $\mathbf{x}$. Then the gradient of $\phi$ exists and is equal to $\mathbf{f}$, that is, $\nabla\phi(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \forall \mathbf{x} \in S$.

**Theorem 10.5** Necessary and Sufficient conditions for a vector field to be a gradient: Let $\mathbf{f}$ be a vector field continuous on an open connected set S in $\mathbb{R}^n$. Then the following three statements are equivalent: (a) $\mathbf{f}$ is the gradient of some potential function in S, (b) the line integral of $\mathbf{f}$ is independent of the path in S, (c) the line integral of $\mathbf{f}$ is zero around every piecewise smooth closed path in S.

**Theorem 10.6** Necessary conditions for a vector field to be a gradient: Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a continuously differentiable vector field on an open set S in $\mathbb{R}^n$. If $\mathbf{f}$ is a gradient on S, then the partial derivatives of the components of $\mathbf{f}$ are related by the equations $D_i f_j(\mathbf{x}) = D_j f_i(\mathbf{x})$ for $i, j = 1, 2, \ldots, n$ and every $\mathbf{x}$ in S.

**Theorem 10.7** Assume the differential equation $P(x, y)dx + Q(x, y)dy = 0$ is exact in an open connected set S, and let $\phi$ be a scalar field satisfying $\frac{\partial\phi}{\partial x} = P$ and $\frac{\partial\phi}{\partial y} = Q$ everywhere in S. Then every solution $y = Y(x)$ whose graph lies in S satisfies the equation $\phi[x, Y(x)] = C$ for some constant C. Conversely, if the equation $\phi(x, y) = C$ defines y implicitly as a differentiable function of x, then this function is a solution of the differential equation.

**Theorem 10.8** Differentiation under the integral sign: Let S be a closed interval in $\mathbb{R}^n$ with nonempty interior and let $J = [a, b]$ be a closed interval in $\mathbb{R}^1$. Let $J_{n+1}$ be the closed interval $S \times J$ in $\mathbb{R}^{n+1}$. Write each point in $J_{n+1}$ as $(\mathbf{x}, t), \mathbf{x} \in S$ and $t \in J$. Assume that $\psi$ is a scalar field defined on $J_{n+1}$ such that the partial derivative $D_k\psi$ is continuous on $J_{n+1}$, where $k = 1, 2, \ldots, n$. Define a scalar field $\phi$ on S by the equation $\phi(\mathbf{x}) = \int_a^b \psi(\mathbf{x}, t)dt$. Then the partial derivative $D_k\phi$ exists at each interior point of S and is given by the formula $D_k\phi(\mathbf{x}) = \int_a^b D_k\psi(\mathbf{x}, t)dt$. In other words, we have $D_k \int_a^b \psi(\mathbf{x}, t)dt = \int_a^b D_k\psi(\mathbf{x}, t)dt$.

**Theorem 10.9** Necessary and sufficient condition for a vector field to be a gradient: Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a continuously differentiable vector field on an open convex set S in $\mathbb{R}^n$. Then $\mathbf{f}$ is a gradient on S iff we have $D_k f_i(\mathbf{x}) = D_j f_k(\mathbf{x})$ for each $\mathbf{x} \in S$ and all $k, j = 1, 2, \ldots, n$.

**Conservative Fields (R Week 6) Corollary 2** Let $D$ be an open set in $\mathbb{R}^n$ and let $f : D \to \mathbb{R}^n$ be a continuous vector field. Then TFAE: (i) $f\nabla\phi$ for some potential function $\phi$, (ii) the line integral of $f$ over piecewise $C^1$ curves in $D$ is path independent (iii) the line integral of $f$ over closed, piecewise $C^1$ curves in $D$ are zero. Any vector field satisfying these is conservative.

**Theorem 11.1** Double integration is linear (not proven): $\forall c_1, c_2 \in \mathbb{R}, \iint_Q [c_1 s(x,y) + c_2 t(x,y)] dxdy = c_1 \iint_Q s(x,y)dxdy + c_2 \iint_Q t(x,y)dxdy$.

**Theorem 11.2** Double integration is additive (not proven): If Q is subdivided into two rectangles $Q_1$ and $Q_2$, then $\iint_Q s(x,y)dxdy = \iint_{Q_1} s(x,y)dxdy + \iint_{Q_2} s(x,y)dxdy$.

**Theorem 11.3** Comparison Theorem (not proven): If $s(x,y) \leq t(x,y)$ for every $(x,y)$ in Q, then $\iint_Q s(x,y)dxdy \leq \iint_Q t(x,y)dxdy$. If $t(x,y) \geq 0$ for all $(x,y) \in Q$, then $\iint_Q t(x,y)dxdy \geq 0$.

**Theorem 11.4** Every function f which is bounded on a rectangle Q has a lower intgral $I_l(f)$ and an upper integral $I_u(f)$ satisfying the inequalities $\iint_Q s \leq I_l \leq I_u \leq \iint_Q t$ for all step functions s and t with $s \leq f \leq t$. The function f is integrable on Q iff its upper and lower integrals are equal, in which case we have $\iint_Q f = I_l(f) = I_u(f)$.

**Theorem 1 (R Week 4)** Every continuous function $f$ on a closed rectangular box R is integrable.

**Small Span Theorem (R Week 4)** For every $\epsilon > 0$, there exists a partition $P = \{S_j\}_{j=1}^r$ of $R$ such that $span_f(S_j) < \epsilon$ for each $j \in \{1, \ldots, r\}$.

**Theorem 11.5 (Fubini's Theorem)** Let f be defined and boundeed on a rectangle $Q = [a,b] \times [c,d]$ and assume that f is integrable on Q. For each fixed y in $[c,d]$, assume that the one-dimensional integral $\int_a^b f(x,y)dx$ exists, and denote its value by $A(y)$. If the integral $\int_c^d A(y)dy$ exists it is equal to the double integral $\iint_Q f$. In other words, $\iint_Q f(x,y)dxdy = \int_c^d [\int_a^b f(x,y)dx]dy$. Note that if f is non-negative, then this integral is equal to the volume of the ordinate set of f over Q.

**Theorem 11.6** Integrability of continuous functions: If a function f is continuous on a rectangle $Q = [a,b] \times [c,d]$, then f is integrable on Q. Moreover, the value of the integral can be obtained by iterated integration $\iint_Q f = \int_c^d [\int_a^b f(x,y)dx]dy = \int_a^b [\int_c^d f(x,y)dy]dx$.

**Integration on compact regions: Theorem 6 (R Week 4)** Let Z be a compact subset of $\mathbb{R}^n$ such that the boundary of Z has content zero. Then any function $f$ on Z which is continuous on Z is integrable over Z.

**Theorem 11.7** Let f be defined and bounded on a rectangle $Q = [a,b] \times [c,d]$. If the set of discontinuities of f in Q is a set of content zero then the double integral $\iint_Q f$ exists.

**Theorem 11.8** Let $\phi$ be a real-valued function that is continuous on an interval $[a,b]$. Then the graph of $\phi$ has content zero.

**Theorem 11.9** Let S be a region of type I, between the graphs of $\phi_1$ and $\phi_2$. Assume that f is defined and bounded on S and that f is continuous on the interior of S. Then the double integral $\iint_S f$ exists and can be evaluated by repeated one-dimensional integration: $\iint_S f(x,y)dxdy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\right] dx$. This works for a region of type II also, but just reverse the order of integration.

**Pappus' Theorem** Consider a plane region Q lying between the graphs of two continuous functions f and g over an interval $[a,b]$, where $0 \leq g \leq f$. Let S be the solid of revolution generated by rotating Q about the x-axis. Let $a(Q)$ denote the area of Q, $v(s)$ the volume of S and $\bar{y}$ the y-coordinate of the centroid of Q. As Q is rotated to generate S, the centroid travels along a circle of radius $\bar{y}$. Pappus' theorem states that the volume of S is equal to the circumference of this circle multipled by the area of Q: $v(S) = 2\pi\bar{y}a(Q)$.

**Jordan Curve Theorem (R Week 7)** Let C be a Jordan curve in $\mathbb{R}^2$. Then there exists connected open sets $U, V$ in the plane such that (i) $U, V, C$ are pairwise mutually disjoint, and (ii) $\mathbb{R}^2 = U \cup V \cup C$.

**Theorem 11.10: Green's Theorem** Let P and Q be scalar fields that are continuously differentiable on an open set S in the xy-plane. Let C be a piecewise smooth Jordan curve, and let R denote the union of C and its interior. Assume R is a subset of S. Then we have the identity $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C (Pdx + Qdy)$ where the line integral is taken around C in the counterclockwise direction. Note that this is equivalent to the two formulae: $\iint_R \frac{\partial Q}{\partial x} dxdy = \oint_C Qdy$ and

$-\iint_R \frac{\partial P}{\partial y} dxdy = \oint_C P dx$, by taking $P = 0$ or $Q = 0$ respectively.

**Green's Theorem using Curl (R Week 7a)** Consider a plane region $\Phi$ with boundary as a piecewise $C^1$ Jordan curve C, with a $C^1$ vector field $g = (P, Q)$ on an open set $D$ containing $\Phi$. Then $\iint_\Phi (\nabla \times f) \cdot k dxdy = \oint_C P dx + Q dy$.

**Area expressed as a line integral** $a(R) = \frac{1}{2} \int_a^b \begin{vmatrix} X(t) & Y(t) \\ X'(t) & Y'(t) \end{vmatrix} dt$.

**Isoperimetric Inequality (Wirtinger Inequality, R Week 7)** $4\pi A \leq L^2$, $L = \oint_C ds$ is the length of a Jordan curve C.

**Theorem 11.11** Let $f(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field that is continuously differentiable on an open simply connected set S in the plane. Then $f$ is a gradient on S if and only if we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on S.

**Theorem 11.12** Green's Theorem for Multiply Connected Regions. Let $C_1, \ldots, C_n$ be $n$ piecewise smooth Jordan curves having the following properties: (a) No two of the curves intersect. (b) The curves $C_2, \ldots, C_n$ all lie in the interior of $C_1$. (c) Curve $C_i$ lies in the exterior of curve $C_j$ for each $i \neq j, i > 1, j > 1$. Let R denote the region which consists of the union of $C_1$ with that portion of the interior of $C_1$ that is not inside any of the curves $C_2, C_3, \ldots, C_n$. Let P and Q be continuously differentiable on an open set S containing R. Then we have the following identity: $\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{C_1} (P dx + Q dy) - \sum_{k=2}^n \oint_{C_k} (P dx + Q dy)$.

**Theorem 11.13** Invariance of a line integral under deformation of the path. Let P and Q be continuously differentiable on an open connected set S in the plane, and assume that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on S. Let $C_1$ and $C_2$ be two piecewise smooth Jordan curves lying in S and satisfying the following conditions: (a) $C_2$ lies in the interior of $C_1$ (b) Those points inside $C_1$ which lie outside $C_2$ are in S. Then we have $\oint_{C_1} P dx + Q dy = \oint_{C_2} P dx + Q dy$ where both curves are traversed in the same direction.

**Transforming Double Integrals** Consider $X(u, v)$ and $Y(u, v)$ in $C^1$ on S. Let T be the set of points in the $uv$ plane that is mapped on to the $xy$ plane. The double integral can be written as $\iint_S f(x, y) dxdy = \iint_T f[X(u, v), Y(u, v)] |J(u, v)| dudv$, where $J(u, v)$ is the Jacobian determinant. If $J(u, v) = 0$ at a particular point, that point is called a singular point. The transformation formula is value when the singular points form a set of content zero.

**Linear Transformation** Consider a linear transformation $x = Au + Bv, y = Cu + Dv$, $A, B, C, D$ constants. Then $J(u, v) = AD - BC$, and for the linear transformation to have an inverse, $J(u, v) \neq 0$. The transformation formula is $\iint_S f(x, y) dxdy = |AD - BC| \iint_T f(Au + Bv, Cu + Dv) dudv$.

**Cross Product Lemma 1 (R Week 7a)** (a) $v \times v' = -v' \times v$, (b) $i \times j = k, j \times k = i, k \times i = j$, (c) $v \cdot (v \times v') = v' \cdot (v \times v') = 0$.

**Curl Proposition 1 (R Week 7a)** Let $h$ be a $C^2$ scalar field and let $f$ be a $C^2$ vector field. Then (a) $\nabla \times (\nabla h) = 0$ (b) $\nabla \cdot (\nabla \times f) = 0$.

**Zero curl is conservative (R Week 7a)** Let $g : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, D open and simply connected, $g = (P, Q)$ being a $C^1$ vector field. Set $f(x, y, z) = g(x, y)$ for all $(x, y, z) \in \mathbb{R}^3$ with $(x, y) \in D$. Suppose $\nabla \times f = 0$. Then $g$ is conservative on $D$.

**Change of variables in an n-fold integral** Define new variables $u_1, \ldots, u_n$ such that $x_1 = X_1(u_1, \ldots, u_n), \ldots, x_n = X_n(u_1, \ldots, u_n)$. If this is a one-to-one continuously differentiable mapping on T with Jacobian never zero, the transformation formula is $\int_S f(x) dx = \int_T f(X(u)) |det DX(u)| du$, where $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_n)$.

**Transformation Formula (R Week 8)** Let $D$ be a bounded open set in $\mathbb{R}^n$, $\phi : D \to \mathbb{R}^n$ a $C^1$ one-to-one mapping with Jacobian determinant $\det(D\phi)$ non-vanishing everywhere on $D$. Let $D* = \phi(D)$, and let $f$ be an integrable function on $D*$. Then $\int \cdots \int_D f(\phi(u)) |\det D\phi(u)| du_1 \cdots du_n = \int \cdots \int_{D*} f(x) dx_1 \cdots dx_n$.

**Area cosine principle** If a region $S$ in one plane is projected onto a region $T$ in another plane, making an angle $\gamma$ with the first plane, then the area of $T$ is $\cos \gamma$ times the area of $S$.

**Implicit area** Suppose S is given by an implicit representation $F(x, y, z) = 0$. If $S$ can be projected in a one-to-one fashion on the xy-plane, the equation $F(x, y, z) = 0$ defines $z$ as a function of $x$ and $y$, say $z = f(x, y)$ and the partial derivatives are $\frac{\partial f}{\partial x} = -\frac{\partial F / \partial x}{\partial F / \partial z}$ and $\frac{\partial f}{\partial y} = -\frac{\partial F / \partial y}{\partial F / \partial z}$ for those points at which $\frac{\partial F}{\partial z} \neq 0$. Hence we have that $a(S) = \iint_T \frac{\sqrt{(\partial F / \partial x)^2 + (\partial F / \partial y)^2 + (\partial F / \partial z)^2}}{|\partial F / \partial z|} dxdy$.

**Explicit parametrization (R Week 8)** Let $\Phi$ be a surface in $\mathbb{R}^3$ parametrized by a $C^1$, 1-1 function $\phi : T \to \mathbb{R}^3$, $\phi(u,v) = (u, v, h(u,v))$, which means that $\Phi$ is the graph of $z = h(x,y)$. Then for any integrable scalar field $f$ on $\Phi$, we have $\iint_\Phi f dS = \iint_T f(u,v,h(u,v))\sqrt{\left(\frac{\partial h}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2 + 1} dudv$.

**Theorem of Pappus (Surface of Revolution)** The surface of revolution obtained by rotating a plane curve of length L about an axis in the plane of the curve has area $2\pi Lh$, where $h$ is the distance from the centroid of the curve to the axis of rotation. If the equation of the curve on a plane was $z = f(x), a \leq x \leq b, a \geq 0$, then the surface of revolution when the curve is rotated in the z-axis is $a(S) = 2\pi \int_a^b u\sqrt{1 + [f'(u)]^2} du$.

**Theorem 12.1** Let $r$ and $R$ be smoothly equivalent functions related by $R(s,t) = r[G(s,t)]$ where $G = Ui + Vj$ is a one-to-one continuously differentiable mapping of a region B in the $st$-plane onto a region A in the $uv$-plane given by $G(s,t) = U(s,t)i + V(s,t)j, (s,t) \in B$. Then we have $\frac{\partial R}{\partial s} \times \frac{\partial R}{\partial t} = \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) \frac{\partial(U,V)}{\partial(s,t)}$ where the partial derivatives $\partial r/\partial u$ and $\partial r/\partial v$ are evaluated at the point $(U(s,t), V(s,t))$.

**Theorem 12.2** Let $r$ and $R$ be smoothly equivalent functions. If the surface integral $\iint_{r(A)} f dS$ exists, the surface integral $\iint_{R(B)} f dS$ also exists and we have $\iint_{r(A)} f dS = \iint_{R(B)} f dS$.

**Volumes and Surface Areas of n-spheres (Wiki)** The n-sphere is the set of points in $n+1$ space that are a fixed distance from a particular point. Hence a 2-sphere is the usual surface of a sphere in 3 dimensions. $V_0 = 1, S_0 = 2, V_{n+1} = S_n/(n+1), S_{n+1} = 2\pi V_n$ for a unit n-sphere. Or in closed form, $S_{n-1} = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} R^{n-1}$ and $V_n(R) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} R^n$.

**Volume of parallelpiped (R Week 8)** If $v_1, \ldots, v_n$ are linearly independent vectors in $\mathbb{R}^n$, then the volume of the parallelpiped $P$ spanned by the vectors is $vol(P) = |\det(v_{ij})|$, where $v_{ij}, j = 1, \ldots, n$ are the coordinates of $v_i$. Note that $(v_{ij}) \cdot (v_{ij})^t$ is the symmetric matrix with entries $< v_j, v_i >$, the inner product of the $j$th and $i$th vector. Hence $vol(P) = \sqrt{det(< v_i, v_j >)}$.

**Volume integral of scalar field on k-fold parametrization (R Week 8)** Consider a k-fold parametrization $\Phi \subseteq \mathbb{R}^n$ and an integrable scalar valued function $f$ over $\Phi$. Then $\iint_\Phi f dV = \iint_T f(u)\sqrt{\det\left(< \frac{\partial \phi}{\partial u_i}(u), \frac{\partial \phi}{\partial u_j}(u) >\right)} du_1 \cdots du_k$.

**Lemma 1 (R Week 9)** Let $A = (a_1, a_2, a_3), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3)$. Then $A \cdot (B \times C) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$.

**Theorem 12.3: Stokes' Theorem** Assume that S is a smooth simple parametric surface, say $S = r(T)$, where $T$ is a region in the $uv$-plane bounded by a piecewise smooth Jordan curve $\Gamma$. Assume also that $r$ is a one-to-one mapping whose components have continuous second-order partial derivatives on some open set containing $T \cup \Gamma$. Let $C$ denote the image of $\Gamma$ under $r$, and let $P, Q, R$ be continuously differentiable scalar fields on $S$. Let the curve $\Gamma$ be traversed in the positive (counterclockwise) direction and let the curve C be traversed in the direction inherited from $\Gamma$ through the mapping function $r$. Then we have:

$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_C P dx + Q dy + R dz$$

We can also write this as $\iint_S (\nabla \times F) \cdot n dS = \int_C F \cdot d\alpha$.

**Theorem 12.4** Let $F = (P, Q, R)$ on a continuously differentiable vector field on an open convex set S in 3-space. Then $F$ is a gradient on S iff we have $curl F = 0$ on S.

**Properties of the curl and divergence** $\nabla \times (\nabla \phi) = 0, \nabla \cdot (\nabla \times \phi) = 0$ for every scalar field with continuous second order mixed partial derivatives ($C^2$).

**Theorem 12.5** Let $F$ be continuously differentiable on an open interval $S$ in 3-space. Then there exists a vector field $G$ such that $\nabla \times G = F$ iff $\nabla \cdot F = 0$ everywhere in S.

**Theorem 12.6: Divergence Theorem** Let $V$ be a solid in 3-space bounded by an orientable closed surface $S$, and let $n$ be the unit outer normal to S. If $F$ is a continuously differentiable vector field defined on V, we have $\iiint_V (\nabla \cdot F) dxdydz = \iint_S F \cdot n dS$. Writing $F = (P, Q, R)$ and $n = \cos\alpha i + \cos\beta j + \cos\gamma k$, we re-write this as $\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma) dS$.

**Theorem 2: Gauss Divergence Theorem (R Week 9)** Let $V$ be a region in $\mathbb{R}^3$ with boundary $\Phi$, a closed surface, oriented by choosing the unit outward normal $n$ to $\Phi$. Let $F = (P, Q, R)$ be a $C^1$ vector field on $V$. Then we have $\iiint_V (\nabla \cdot F) dxdydz = \iint_\Phi F \cdot ndS$. In the notation of exterior differential calculus, $\iiint_V \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz = \iint_\Phi P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$.

**Theorem 12.7** Let $V(t)$ be a solid sphere of radius $t > 0$ with center at a point $a$ in 3-space, and let $S(t)$ denote the boundary of $V(t)$. Let $F$ be a vector field that is continuously differentiable on $V(t)$. Then if $|V(t)|$ denotes the volume of $V(t)$, and if $n$ denotes the unit outer normal of $S$, we have $\nabla \cdot F(a) = \lim_{t \to\ 0} \frac{1}{|V(t)|} \iint_{S(t)} F \cdot ndS$.

**Curl Alternative Definition (Equation 12.61)** $\nabla \times F(a) = \lim_{t \to 0} \frac{1}{|V(t)|} \iint_{S(t)} n \times F dS$, where $V(t)$ is a solid sphere of radius $t > 0$ centered at a point $a$ in 3-space and $S(t)$ is the boundary of $V(t)$. $n$ is the unit outer normal of $S$.

**Curl Alternative Definition (Equation 12.62)** $n \cdot (\nabla \times F(a) = \lim_{t \to 0} \frac{1}{|S(t)|} \oint_{C(t)} F \cdot d\alpha$. where $\alpha$ is the function that traces out $C(t)$ in a direction that appears to be counterclockwise when viewed from the tip of $n$.