# Ma2 Class Notes

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# Chapter 1

# Week 1

## 1.1 Monday, 29 Sept 2014

**Features of ODE problems:** Deterministic, finite dimensional, differentiable.

**Differentiable manifold:** Manifold that is locally similar to a plane (the tangent plane).

**General form:** Unknown is a vector dependent on time: $\vec{x}(t)$. Write $\frac{d^k\vec{x}(t)}{dt^k} = f\left(t, \vec{x}(t), \frac{d\vec{x}(t)}{dt}, \cdots, \frac{d^{k-1}\vec{x}(t)}{dt^{k-1}}\right)$ for some function $f : \mathbb{R}^{kn+1} \to \mathbb{R}^n$ in a vector space with dimension $n$.

**More generally,** since the highest derivative may not be able to be expressed explicitly, we write the implicit form: $F\left(t, \vec{x}(t), \cdots, \frac{d^k\vec{x}(t)}{dt^k}\right) = 0$ for the function $F : \mathbb{R}^{(k+1)n+1} \to \mathbb{R}^n$.

**Order:** Call an equation of order $k$ if the highest derivative in the general equation is the $k$th.

## 1.2 Wednesday, 1 Oct 2014

**Simplest possible DE** Consider $\frac{dx(t)}{dt} = f(t)$. We have some requirements for the assigned function $f(t)$: $f : [a, b] \to \mathbb{R}$ and is continuous for $a \leq t \leq b$. Solve using usual integration plus constant.

**Another case:** Consider $\frac{dx(t)}{dt} = f(x(t))$. Assume $f : [a, b] \to \mathbb{R}$ and is continuous on the interval. Also assume $f(x) \neq 0 \forall x \in [a, b]$. Rewrite equation as $\frac{1}{f(x)}\frac{dx}{dt} = 1$. Integrate both sides in the variable $t$.

**General situation for separation for variables:** $\frac{dx}{dt} = f(x)g(t)$, and assume that $f$ and $g$ are continuous, and that $f(x) \neq 0$ on that interval.

**Integrating Factors:** Recall from calculus that the derivative of the product of two functions $\frac{d}{dt}(f(t)g(t)) = \frac{df(t)}{dt}g(t) + \frac{dg(t)}{dt}f(t)$.

## 1.3 Thursday, 2 Oct 2014, Recitation

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**Possible alternative: 1pm 115 Beckmann, 1pm 102 Steele**

## 1.4 Friday, 3 Oct 2014

**Existence and uniqueness** Consider $x' + p(t)x = q(t)$. Assumptions: $p(t)$ and $q(t)$ are continuous on an open interval $t \in (a, b)$ that contains $t_0$. Also suppose that the initial conditions $x(t_0) = x_0$ is specified. Then there is a unique solution $x(t)$ such that $x(t_0) = x_0$. This is true in the case of first order linear equations by explicit construction using integrating factors.

# Chapter 2

# Week 2

## 2.1 Monday 6 Oct 2014

**Existence and Uniqueness Theorem** Consider $y' = f(t, y(t))$ and initial conditions $t_0$ and $y(t_0) = y_0$. We require that $f(t, y)$ is continuous and $\frac{\partial f}{\partial y}$ is continuous, and both continuity conditions hold for all $t$ in an interval $t \in (\alpha, \beta)$ and $y \in (\gamma, \delta)$ such that $(t_0, y_0) \in (\alpha, \beta) \times (\gamma, \delta)$. We now want to find some function $y(t) = \phi(t)$ such that $\begin{cases} \phi'(t) = f(t, \phi(t)) \\ \phi(t_0) = y_0 \end{cases}$. Re-think this problem as a "fixed point problem", where we want to find a transformation $T$ acting on the functions of $t$: $\psi(t) \mapsto T(\psi(t))$ for which $\exists \phi(t)$ such that $T(\phi(t)) = \phi(t)$. Motivation: for fixed point problems, there are existence and uniqueness results that are already available. We note that a differential equation can be written equivalently as an integral equation: $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases} \iff y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$. Now the integral equation is exactly the fixed point problem for the transformation $T(\psi(t)) = y_0 + \int_{t_0}^{t} f(s, \psi(s))ds$.

**Existence and Uniqueness of Fixed Point Problems** If we have a transformation $T : X \mapsto X$, and (1) $T$ is continuous, and (2) $X$ is a complete metric space, (metric space=distance between all members in a set is defined, complete=every Cauchy sequence of points in the set converges to a point in the set) and (3) $T$ contracts distances, then there is a unique point $x \in X$ such that $T(x) = x$. So $T$ has a unique fixed point in $X$. It also follows that we can approximate the solution to $T(x) = x$ by starting from any point $x_0 \in X$ by repeatedly applying $T$ (since $T$ contracts distances) to obtain a sequence of points $\{T^{(n)}(x_0)\}$ to converge at the fixed point.

**Applying the Fixed Point Theorem to the DE** We note that the property of $\frac{\partial f}{\partial y}$ being continuous on $\mathbb{R}$ implies that $|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$ (Lipschitz continuity of $f(x, y)$ in y). This comes from the mean value theorem, which states that there is some $y_t$ such that $y_1 \leq y_t \leq y_2$ such that $\frac{\partial f(t,y_t)}{\partial y} = \frac{f(t,y_2)-f(t,y_1)}{y_2-y_1}$. Hence we need to show that $\left|\frac{\partial f(t,y_t)}{\partial y}\right|$ is bounded. We know that $\frac{\partial f}{\partial y}$ is continuous on $(\alpha, \beta) \times (\gamma, \beta)$, hence on any closed subinterval $[\alpha+\epsilon, \beta-\epsilon] \times [\gamma+\epsilon, \delta-\epsilon]$ inside the open rectangle it achieves both a minimum and a maximum. Hence $\left|\frac{\partial f(t,y_t)}{\partial y}\right|$ is bounded by $K$. $|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$ will be used to show that $T$ contracts distances.

**Measurement of distances between functions** If we have $\psi_1(t)$ and $\psi_2(t)$ being two functions, the distance between the two functions is $distance(\psi_1(t), \psi_2(t)) = \sup_{\alpha+\epsilon \leq t \leq \beta-\epsilon} |\psi_1(t) - \psi_2(t)|$.

## 2.2 Wednesday 8 Oct 2014

**Distance function** Consider $d : X \times X \to \mathbb{R}^+$ that satisfies (1) $d(x, y) = d(y, x), \forall x, y \in X$, (2) $d(x, y) = 0 \iff x = y$ and (3) satisfies the triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$. For instance, the distance function on the real line $d(x, y) = |x - y|$ is one such function. The Euclidean norm is a distance function on $\mathbb{R}^n$.

**Completeness** If we have a sequence of points $\{x_n\}$ in $X$ such that the distances between the points gets smaller as $n$ gets larger (Cauchy sequence) $\forall \epsilon > 0, \exists N = N(\epsilon)$ such that $\forall m, n \geq N(\epsilon)$ then $d(x_m, x_n) < \epsilon$. A complete metric space has that every Cauchy sequence in $X$ converges (distance goes to zero) to another point in $X$. Example of a metric space that is not complete: $x_n \to x \in \mathbb{R} \backslash \mathbb{Q}$ is Cauchy in $\mathbb{Q}$ but not convergent in $\mathbb{Q}$ (a sequence that converges to an irrational number by successive approximations using rational numbers does not have a limit in $\mathbb{Q}$).

**Contracts distances** $T : X \to X$ where $X$ is a metric space with distance function $d(x, y)$ contracts distances if $d(T(x), T(y)) \le d(x, y)$. Note that this allows for the distance to remain the same under the transformation $T$.

**Proof of Fixed Point Theorem** We prove this by explicit construction of the fixed point. Start with any point $x_0 \in X$. Consider the sequence $\{x_n\} = T^n(x_0)$. Let the transformation be strongly contracting: $d(T(x), T(y)) \le Kd(x, y)$ where $K < 1$. Then the distance between $x_{n+1}$ and $x_n$, $d(x_{n+1}, x_n) \le Kd(x_n, x_{n-1})$. We can keep going until $d(x_{n+1}, x_n) \le K^n d(x_1, x_0)$. But $K < 1$, so $K^n \to 0$ when $n \to \infty$. WLOG take $m \ge n$. Hence $d(x_m, x_n) \le \sum_{j=n}^{m-1} d(x_{j+1}, x_j)$ using the triangle inequality. Also: $\sum_{j=n}^{m-1} d(x_{j+1}, x_j) \le (K^n + K^{n+1} + \cdots + K^{m-1})d(x_1, x_0)$ and the RHS goes to zero when $n \to \infty$. Hence the sequence is Cauchy. Since the metric space is complete, $x_n \to x \in X$ when $n \to \infty$. This limit is the fixed point, since $x_{n+1} = T(x_n) = x_n$ if $T$ is continuous. Since this is the limit of the same sequence, the fixed point is the limit of the sequence.

**Applying the Fixed Point Theorem** We now consider the transformation $T(\phi(t)) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$. Define $J$ to be a small interval around $t_0$ and define $I$ to be a small interval around $y_0$. Then consider the set of functions that map from $J$ to $I$: $X(J) = \{\phi : J \to I\}$. Recall the distance between functions $d(\phi_1(t), \phi_2(t)) = \sup_{t \in J} |\phi_1(t) - \phi_2(t)|$. We claim that $X(J)$ with this distance metric is complete (non-trivial and not proven here). Now we need to show that $T$ is a contraction. We use the hypothesis that $\frac{\partial f}{\partial y}$ is continuous. This implies that $|f(t, y_1) - f(t, y_2)| \le C|y_1 - y_2|$ by the mean value theorem, where $C$ is the supremum of $\frac{\partial f}{\partial y}$ on the interval. We now examine the $\sup \left| \int_{t_0}^{t} f(s, \phi_1(s))ds - \int_{t_0}^{t} f(s, \phi_2(s))ds \right| = d(T(\phi_1), T(\phi_2))$. By the previous mean value theorem result, we have that this is less or equal to $\int_{t_0}^{t} \sup_t |f(s, \phi_1(s)) - f(s, \phi_2(s))| \, ds \le C \sup_t |\phi_1(t) - \phi_2(t)| \int_{t_0}^{t} ds$. But $\int_{t_0}^{t} ds = length(J)$. Hence we have $d(T(\phi_1), T(\phi_2)) \le C \times length(J) d(\phi_1, \phi_2)$.

## 2.3 Thursday 9 Oct 2014 Recitation

**Tips to find integrating factors** If $\frac{M_y - N_x}{N} = f(x)$, then use integrating factor $e^{\int f(x)dx}$. If $\frac{M_y - N_x}{M} = f(y)$ then use integrating factor $e^{\int f(y)dy}$. If $\frac{M_y - N_x}{M - N} = f(x + y)$ then use integrating factor $e^{\int f(x+y)dy}$?.

## 2.4 Friday 10 Oct 2014

**Sufficiency of condition on exactness** Define $M(x, y) + N(x, y)y' = 0$. Then we have that a necessary condition for exactness is $M_y = N_x$. But to determine if this condition is sufficient or not depends on the domain of definition of $M$ and $N$. Suppose $M_y = N_x$ is satisfied. We want to construct a function $\psi(x, y)$ with $\psi_x = M$ and $\psi_y = N$. Start from some $x_0$. Take $Q(x, y) = \int_{x_0}^{x} M(s, y)ds$ and move along the x direction. Taking the derivative of $Q$ with respect to $x$, we just get $\frac{\partial Q(x,y)}{\partial x} = M(x, y)$. Note that we can also add some arbitrary function of $y$ to $Q(x, y)$ and it will also satisfy $Q_x = M(x, y)$. Hence we need to eliminate the ambiguity in $y$. Take the derivative of $Q(x, y)$ with respect to y: $\frac{\partial(Q(x,y)+h(y))}{\partial y} = \frac{\partial Q}{\partial y} + h'(y)$. We want this to equal to $N(x, y)$. Hence we have that $h'(y) = N(x, y) - \frac{\partial Q}{\partial y}(x, y)$. This means that the RHS does not depend on $x$. Hence we can write $N_x - \frac{\partial^2 Q(x,y)}{\partial x \partial y} = N_x - \frac{\partial}{\partial y} Q_x$ after exchanging the order of second derivatives. But we have that $N_x = M_y$, so this equals zero. Hence we can write $h(y) = \int_{y_0}^{y} \left( N(x, s) - \frac{\partial Q}{\partial s}(x, s) \right) ds$. Now this construction has to be independent of the path, so this means that we should be able to deform the path continuously and not change the construction of $\psi$. However, if there are holes in the domain of either $M$ or $N$, then this path independence does not hold.

# Chapter 3

# Week 3

## 3.1 Monday 13 Oct 2014

**Sufficient condition for exactness** If $Domain(M) = Domain(N) = \mathbb{R}^2$, then the necessary condition $M_y = N_x$ is also sufficient.

**What happens when you have holes in the domain?** Write the first order differential as the 1-form $w = M(x,y)dx + N(x,y)dy$. Consider the curve $\gamma$ in the plane parametrised by $t$: $\gamma(t) = (x(t), y(t))$. Then $\int_\gamma \omega = \int_{t_0}^{t} M(x(t), y(t))\frac{dx(t)}{dt} + N(x(t), y(t))\frac{dy(t)}{dt}$. We consider (Case I) the differential equation with $M(x,y) = \frac{x}{x^2+y^2}$ and $N(x,y) = \frac{y}{x^2+y^2}$ such that $M(x,y) + N(x,y)\frac{dy}{dx} = 0$. We note that the domain of $M$ and $N$ is $\mathbb{R}^2\backslash\{(0,0)\}$. Also consider (Case II) $M(x,y) = \frac{-y}{x^2+y^2}, N(x,y) = \frac{x}{x^2+y^2}$. Note that these two cases are fine except around the origin. We look at curves that go around the origin. For the first case, we pick the curve $\gamma = (R\cos t, R\sin t)$. Then, integrating $\int_\gamma \omega = \int_0^{2\pi} \frac{x(t)}{x(t)^2+y(t)^2}\frac{dx(t)}{dt} + \frac{y(t)}{x(t)^2+y(t)^2}\frac{dy(t)}{dt}$. Since $\frac{dx(t)}{dt} = -R\sin t, \frac{dy(t)}{dt} = R\cos t$, we can obtain $\int_\gamma \omega = \int_0^{2\pi} -\cos t \sin t dt + \sin t \cos t dt = 0$. Hence the integral for any curve that circles around the origin for the first DE case is zero. For the second case, we choose the same curve and obtain: $\int_\gamma \omega = \int_0^{2\pi}(\sin^2 t + \cos^2 t)dt = 2\pi$. Hence the contribution of the winding is nonzero for the second case. Every time you go around the origin, you pick up a contribution of $2\pi$, which means that the integral of $\omega_\gamma$ is not path independent. For the first case, the function $\psi(x,y) = \frac{1}{2}\log(x^2+y^2)$ is a well-defined function away from the origin. It also satisfies $\psi_x = M(x,y)$ and $\psi_y = N(x,y)$. However, although the second case satisfies the necessary condition $M_y = N_x$, this condition is no longer sufficient. However, when we try to construct a function that generates $M$ and $N$, such that $\psi(x,y) = arctan(y/x)$ we realise that although $\psi_x = M$ and $\psi_y = N$ some problems arise. The arctangent is defined for $t \in \mathbb{R}$ and takes values in $(-\pi/2, \pi/2)$, and if $t < 0$, $\arctan t + \arctan(1/t) = \frac{-\pi}{2}$, if $t > 0$, $\arctan t + \arctan(1/t) = \frac{\pi}{2}$. Now consider the function $\arctan(y/x) + c$, which is a guess for a possible solution of the second DE case. This solution is not defined when $x = 0$, since there is a $y/x$. Hence we have two cases: $x > 0$ and $x < 0$. For the former case, $x > 0$ so we consider what happens when $x \to 0$ from $x > 0$ and $y \to y_0 > 0$. Then $\psi(x,y) \to \pi/2 + c$. If we let $x \to 0$ from $x > 0$ with $y \to y_0 < 0$, then $\psi(x,y) \to -\pi/2 + c$. However, if we approach $x \to 0$ from $x < 0$ and $y$ positive, $\psi(x,y) \to -\pi/2 + k$ for some constant $k$ and if we approach with $y$ negative, $\psi(x,y) \to \pi/2 + k$. We note that we cannot choose a $c$ and $k$ such that the solution is continuous along the $y$ and $x$ axes. We must have that $\psi$ is discontinuous on a line emanating from the origin towards infinity. To understand this, we look at the polar coordinates in the plane $(x,y) \to (r,\theta)$. For $r \neq 0$, $\theta$ is well-defined and we can write $\theta = \begin{cases} \arctan(y/x), x > 0 \\ \pi/2 - \arctan(x/y), y > 0 \\ -\pi/2 - \arctan(x/y), y < 0 \\ \text{undefined}, x = y = 0 \end{cases}$. Note further that $d\theta = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy$.

## 3.2 Wednesday 15 Oct 2014

**Autonomous Differential Equation** Consider $y' = f(y)$ without explicit dependence on time. General strategy: (1) Look for equilibrium solutions. (i.e. $f(y) = 0$). (2) Look at the sign of $f(y)$. (3) Examine the convexity of the function by examining the signs of $f'(y)$ and $f(y)$ since $y''(t) = f'(y)y' = f'(y)f(y)$ hence the convexity depends on the sign of the product of $f'(y)$ and $f(y)$.

### 3.3 Friday 17 Oct 2014

**Second Order Equations** Write $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$. If $f$ is linear, we can write $\frac{d^2y}{dt^2} = a(t)\frac{dy}{dt} + b(t)y + c(t)$.

**Initial conditions** We require $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

**Superposition** For a linear and homogenous second order equation, we have the superposition principle: Linear combinations of solutions are also solutions. Note that this doesn't work if we have solutions to the non-homogenous equation. If $y(t) = \alpha y_1(t) + \beta y_2(t)$ is a solution, then we can impose the initial conditions to require that $\alpha y_1(t_0) + \beta y_2(t_0) = y_0$ and $\alpha y_1'(t_0) + \beta y_2'(t_0) = y_0'$. This is a linear system that we can solve for $\alpha$ and $\beta$. Write $\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$ which has a solution provided the determinant of the matrix is non-zero. Hence we require that $y_1(t_0)y_2'(t_0) \neq y_1'(t_0)y_2(t_0)$.

**Wronskian** This is the two-dimensional case of the Wronskian. Write $W(y_1, y_2)(t)$, a function of $t$, defined as the determinant of the matrix $\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$. The condition to solve for $\alpha$ and $\beta$ is that $W(y_1, y_2)(t_0) \neq 0$. More generally, the Wronskian of $n$ functions is an nxn square matrix with the functions in the first row, and the $(n-1)st$ derivatives in the rows subsequent to it. Note that a matrix determinant is non-zero iff the column vectors are linearly independent. Define the two vectors $\underline{x}_1(t) = (x_1(t), v_1(t))^T, \underline{x}_2(t) = (x_2(t), v_2(t))^T$. Then $W(\underline{x}_1(t), \underline{x}_2(t)) = x_1(t)v_2(t) - x_2(t)v_1(t)$. If the two vectors are linearly dependent, then there are two numbers $a, b \in \mathbb{R}$ such that $a\underline{x}_1(t) + b\underline{x}_2(t) = 0$. Note that $a$ and $b$ do not change with $t$ hence the Wronskian vanishes for all $t$. In other words, vanishing Wronskian for all $t$ is a necessary condition for the linear dependence of the two vectors. It is not a sufficient condition! Counterexample: take $\underline{x}_1(t) = (t^2, 2t)^T$ and $\underline{x}_2(t) = (t|t|, 2|t|)$. Then the Wronskian is identically zero. But these two vectors are not linearly dependent because when $t > 0$, we can choose $a = -b$ such that $a\underline{x}_1(t) + b\underline{x}_2(t) = 0$. But when $t < 0$ then we require $a = b$ instead. But the only way that this is satisfied for all $t$ is for $a = b = 0$. However the contrapositive is true: If the Wronskian is non-zero for at least one $t$ then $\underline{x}_1(t)$ and $\underline{x}_2(t)$ are linearly independent.

**Constant coefficient homogenous second order equations** Write $ay'' + by' + cy = 0$. We can try an exponential function $y = e^{rt}$. We plug this into the equation to get $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$. We hence have an equation for $r$ which is $ar^2 + br + c = 0$. Three possibilities: (1) the roots of this equation are both real and different, (2) the roots of this equation are real and equal, and (3) the roots are complex but complex conjugates of each other. For case (1), write $y(t) = \alpha e^{r_+t} + \beta e^{r_-t}$. We calculate the Wronskian of the solutions $y_1(t) = e^{r_+t}$ and $y_2(t) = e^{r_-t}$ to be $(r_- - r_+)e^{(r_++r_-)t}$. We just require the Wronskian to be non-zero at one point to know that the solutions are linearly independent. We just evaluate the Wronskian at $t = 0$, and obtain that it is non-zero. Substituting the initial conditions, we have that $\alpha + \beta = y_0$ and $\alpha r_+ + \beta r_- = y_0'$. This determines $\alpha$ and $\beta$ to obtain the unique solution that satisfies the initial condition. For case (2), the solutions are real and equal when the discriminant is equal to zero $b^2 - 4ac = 0$. Call $r_1 = r_2 = r$. Hence we have one solution $y(t) = e^{rt}$. But we need a second solution to have a two-parameter family to solve the initial conditions. We try to guess the second solution by letting the coefficient of $e^{rt}$ vary with $t$. Call this "varying coefficients". So we guess $f(t)e^{rt}$ for some function $f(t)$. Now for this to be a solution of the equation, we plug it into the differential equation to obtain: $a(f''(t)e^{rt} + 2f'(t)re^{rt} + f(t)r^2e^{rt}) + b(f'(t)e^{rt} + f(t)re^{rt}) + cf(t)e^{rt} = 0$. Simplifying, $a(f'' + 2f'r + fr^2) + b(f' + rf) + cf = 0$. Rearranging, $a'' + 2af'r + bf' + (ar^2 + br + c)f = 0$. But we know that $r$ is a root of $ar^2 + br + c = 0$. Hence we have that $af'' + (2ar + b)f' = 0$. But we have that $-b/2a = r$ from the quadratic equation, so $2ar + b = 0$. Hence we just have $f'' = 0$, or $f' = c$ constant. Hence we have $f(t) = c_1t + c_2$ for two constants. Hence we have that the solution we guessed will be a solution of the form $(c_1t + c_2)e^{rt}$. Hence the superposition of two solutions will be $y(t) = \alpha te^{rt} + \beta e^{rt}$. We hence have to check that the Wronskian of the two solutions: $W(e^{rt}, re^{rt}) \neq 0$ hence the solutions are linearly independent. For case (3), we let $r_1$ and $r_2$ be complex conjugates of each other, which happens when the discriminant is negative. Let $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$. Hence when we write $e^{rt}$ we can write it as $e^{\lambda t}(\cos \mu t \pm \sin \mu t)$. Combining the two terms, we can write $y_1(t) = e^{\lambda t}\cos(\mu t)$ and $y_2(t) = e^{\lambda t}\sin(\mu t)$ such that their linear superposition is a general solution of the equation. We note that both $y_1$ and $y_2$ are linearly independent because their Wronskian is $e^{2\lambda t}\mu$ which is non-zero because $\mu \neq 0$ and the exponential is never zero. Hence the general equation is $\alpha e^{\lambda t}\cos(\mu t) + \beta e^{\lambda t}\sin(\mu t)$.

# Chapter 4

# Week 4

## 4.1 Monday 20 Oct 2014

**Existence and Uniqueness for Second Order Differential Equations** Also applies for higher order DEs. For a second order equation $y'' = f(t, y, y')$ with some initial conditions $y(t_0) = y_0, y'(t_0) = y_0'$. We transform this into a first order problem. We introduce a new variable $v(t) = y'(t)$. Hence we have a system of equations $y'(t) = v(t), v'(t) = f(t, y, v)$ which is a first order system in two variables. We hence seek the unknown function vector $\mathbf{Y}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}$. The derivative of this vector is some function $\mathbf{Y}' = \mathbf{F}(t, \mathbf{Y})$ where $\mathbf{F}(t, \mathbf{Y}) = \begin{pmatrix} v(t) \\ f(t, y(t), v(t)) \end{pmatrix}$. We also have the initial conditions $\mathbf{Y}_0 = \begin{pmatrix} v(t_0) \\ f(t_0, y(t_0), y'(t_0)) \end{pmatrix}$. For n-derivatives, we have $y^{(n)} = f(t, y, y', y'', \cdots, y^{(n-1)})$. So we can transform this into the vector-valued function $\mathbf{Y}'(t) = \mathbf{F}(t, \mathbf{Y})$ with $\mathbf{Y}(t_0) = \mathbf{Y}_0$ when we let $\mathbf{Y}(t) = \begin{pmatrix} y(t) \\ v'(t) \\ \vdots \\ v_{n-1}(t) \end{pmatrix}$ with $y' = v_1, y'' = v_2, \ldots, y^{(n-1)} = v_{n-1}, y^{(n)} = f(t, y, v_1, v_2, \cdots, v_{n-1})$. The first-order uniqueness and existence equations proved earlier in the course can be fully generalised by replacing the absolute value by the norm of the $n$-vector. Hence we have the existence and uniqueness of all the highest order equations without having to prove everything again. Note that for linear homogenous equations of order $n$, then we can just look for n solutions $y_1(t), \ldots, y_n(t)$ and take linear combinations of these n equations to obtain the general solution. To verify that these n equations are linearly independent, we just check their Wronskian (now need to evaluate $(n-1)$st derivatives), and ensure that it is non-zero at at least one point. We note that if the solutions have the form $e^{\alpha_i t}$ where all the $\alpha_i$ are not equal, then the determinant can be calculated to be $e^{\alpha_1 + \ldots + \alpha_n}$ multiplied by the Vandermode determinant $\prod_{1 \leq i < j \leq n}(\alpha_i - \alpha_j)$. Hence whenever $\alpha_i$ are not all equal, the determinant does not vanish and we have linearly independent equations.

## 4.2 Wednesday 22 Oct 2014

**Inhomogeneous second order linear equations** Consider $y'' + p(t)y' + q(t)y = g(t)$. Observe that if there are two solutions $Y_1$ and $Y_2$ of the non-homogenous equation, then the difference of the two solutions is a solution to the homogeneous equation. Hence if you know the solution to the homogeneous equation, such as $c_1 y_1(t) + c_2 y_2(t)$, then you can find the general solution to the inhomogeneous equation by adding on a particular solution $\phi(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t)$.

**Finding the particular solution** We write $u_1(t)y_1(t) + u_2(t)y_2(t)$ using the "variation of parameters" to write the constants as unknown functions. We note that $y'(t) = u_1 y_1' + u_1' y_1 + u_2' y_2 + u_2 y_2'$ and to simplify, we want to look for solutions that have $u_1' y_1 + u_2' y_2 = 0$. Hence after this simplification, we have $y' = u_1 y_1' + u_2 y_2', y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$. Plugging this into the non-homogeneous equation, we obtain $u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(t)[u_1 y_1' + u_2 y_2'] + q(t)[u_1 y_1 + u_2 y_2]$. Rearranging, we obtain: $u_1(y_1'' + p(t)y_1' + q(t)y_1) + u_2(y_2'' + p(t)y_2' + q(t)y_2) + u_1' y_1' + u_2' y_2'$ which has the first two terms equal to zero because $y_1$ and $y_2$ are solutions to the homogeneous equation. Hence we just want $u_1' y_1' + u_2' y_2' = g(t)$. We hence have a system of equations for $u_1$ and $u_2$: $u_1' y_1 + u_2' y_2 = 0$ and $u_1' y_1' + u_2' y_2' = g(t)$. Note that this is a linear system:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

We note that the 2x2 matrix can be inverted because the determinant is just the Wronskian, which is non-zero. Hence we have:

$$u_1' = \frac{-y_2 g}{W(y_1, y_2)}$$
$$u_2' = \frac{y_1 g}{W(y_1, y_2)}$$

We can integrate these equations to obtain $u_1$ and $u_2$.

$$\phi(t) = -y_1 \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2 \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds + c_1 y_1(t) + c_2 y_2(t)$$

**n-dimensional system** Recall that we can write the matrix differential equation as $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ where $\mathbf{X}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$. and $\mathbf{A}(t)$ is the coefficient matrix $\{a_{ij}(t)\}_{i,j=1,2,\ldots,n}$. Consider the simplest case of constant coefficients, where the matrix $\mathbf{A}$ does not depend on $t$. Observations in the simplest case: the zero vector is always a solution. Also, if you have two solutions $\mathbf{Y}_1(t), \mathbf{Y}_2(t)$ then a linear combination of them is also a solution.

**Solutions of the matrix differential equation** Assume $\mathbf{A}$ is diagonalisable and has real eigenvalues. Then there exists another matrix such that $\mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ is diagonal. Hence we can make a change of variables $\mathbf{Y} = \mathbf{U}^{-1}\mathbf{X}$. Then the system **X**'=**AX** becomes $(\mathbf{UY'})' = \mathbf{AUY}$. But $\mathbf{U}$ has constant coefficients if $\mathbf{A}$ has constant coefficients. Hence we have $\mathbf{Y}' = (\mathbf{U}^{-1}\mathbf{A}\mathbf{U})\mathbf{Y}$ which can be solved because $\mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ is diagonal and hence we have completely uncoupled equations. Write this as $\dot{y}_j(t) = \lambda_j y_j(t), j = 1, 2, \ldots, n$, where $\lambda_j$ is the $j$th eigenvalue. Hence $\mathbf{Y} = diag(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, \ldots, c_n e^{\lambda_n t})$. Then the solution of the original system will be $\mathbf{X} = \mathbf{U}^{-1}\mathbf{Y}$.

**Alternative method of describing the procedure** Let $\lambda_j, v_j = (v_{j,1}, v_{j,2}, \ldots, v_{j,n})^T$ be an eigenvalue and corresponding eigenvector of $\mathbf{A}$. Then we have that $\mathbf{A}v_j = \lambda_j v_j$, and $\mathbf{Y}_j = e^{\lambda_j t} v_j$ is a solution to the matrix differential equation because $\mathbf{A}\mathbf{Y}_j = e^{\lambda_j t}\mathbf{A}v_j = e^{\lambda_j t}\lambda_j v_j = (\frac{d}{dt}e^{\lambda_j t})v_j = \frac{d}{dt}\mathbf{Y}_j$. Hence we can take linear combinations of these solutions to obtain that $\mathbf{Y} = c_1 e^{\lambda_1 t} v_1 + \ldots + c_n e^{\lambda_n t} v_n$ is a general solution to the matrix differential equation. Hence the general procedure is (1) find the eigenvalues of $\mathbf{A}$, (2) find the corresponding eigenvectors of $\mathbf{A}$ (3) build the general solution by taking linear combinations of $e^{\lambda_j} v_j$.

# Chapter 5

# Week 5

## 5.1 Monday 27 Oct 2014

**Matrix differential equation with complex eigenvalues** Note that if the coefficient matrix $A$ contains only real values, then the complex eigenvalues will occur in complex conjugates. Consider the case where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then we have that $\lambda = \pm i$ and the two respective eigenvectors are $(1, i)^T, (1, -i)^T$. Then the solutions can be written as $\vec{z}(t) = \vec{z}_1(t) + \vec{z}_2(t) = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}$. But we have that $\vec{z}'(t) = A\vec{z}(t)$ so by the equality of complex numbers, we have $\Re(\vec{z}'(t)) = \Re(A\vec{z}(t))$ and $\Im(\vec{z}'(t)) = \Im(A\vec{z}(t))$. Hence both the real and imaginary parts of $z$ will satisfy the differential equation. Hence it will suffice to pick the real and imaginary parts of just one complex solution to give two real solutions. Take $\vec{x}_1(t) = \Re(\vec{z}_1(t))$ and $\vec{x}_2(t) = \Im(\vec{z}_1(t))$. Note: verify that the Wronskian for the two solutions does not vanish at at least one point. Then the final solution $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$. If we use the other complex solution, we will get the same general form anyway, maybe with a sign change on the arbitrary coefficient.

**Matrix exponential** Define $e^{\mathbf{M}} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{M}^k$. Define $\mathbf{M}^0 = \mathbf{I}$ to be the identity matrix. Then for the matrix differential equation with $\mathbf{A} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ being diagonal, we can write $e^{\mathbf{A}t} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$. Observe that if we take the derivative with respect to $t$, we just obtain $\frac{d}{dt}e^{\mathbf{A}t} = diag(\lambda_1 e^{\lambda_1 t}, \lambda_2 e^{\lambda_2 t}, \ldots, \lambda_n e^{\lambda_n t}) = \mathbf{A}e^{\mathbf{A}t}$. Hence each column of the matrix $e^{\mathbf{A}t}$ is a solution to $\mathbf{X}' = \mathbf{AX}$. Let $\Phi(t) = e^{\mathbf{A}t}$ be the matrix whose columns are the fundamental set of solutions for $\mathbf{X}' = \mathbf{AX}$. Then we have $\Phi' = \mathbf{A}\Phi$.

**General solution with matrix exponential** We know that if a matrix $\mathbf{A}$ is diagonalisable, then we have some matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{AU}$ is diagonal with eigenvalues along the diagonal. We also realise that $e^{\mathbf{A}t} = \sum_k \frac{t^k}{k!}\mathbf{A}^k$ so $\mathbf{U}^{-1}e^{\mathbf{A}t}\mathbf{U} = e^{\mathbf{U}^{-1}\mathbf{AU}t} = e^{\mathbf{D}t}$ where $\mathbf{D}$ is diagonal. We can bring the matrices into the exponent because we note that $(U^{-1}AU)^2 = U^{-1}AUU^{-1}AU = U^{-1}A^2U$ and for higher powers $(U^{-1}AU)^n = U^{-1}A^nU$ and we can express $e^{\mathbf{A}t} = I + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \ldots$. Hence $\mathbf{U}^{-1}e^{\mathbf{A}t}\mathbf{U} = \sum \mathbf{U}^{-1}\frac{\mathbf{A}^n}{n!}\mathbf{U} = \sum \frac{1}{n!}(\mathbf{U}^{-1}\mathbf{AU})^n = e^{\mathbf{U}^{-1}\mathbf{AU}t}$.

**Non-diagonalizable matrix** Consider the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We note that the eigenvalues are repeated with $\lambda = 1$ and there is only one eigenvector $(1, 0)^T$. We note that taking higher powers of $A$, $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, which can be proven using induction. Then $e^{At} = \begin{pmatrix} e^t & \sum_{n=0}^{\infty} \frac{nt^n}{n!} \\ 0 & e^t \end{pmatrix}$. But we can write $\sum_{n=0}^{\infty} \frac{nt^n}{n!} = t\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = t\sum_{n=0}^{\infty} \frac{t^n}{n!} = te^t$. Hence we have $e^{At} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$. We note that the two columns are two independent functions. Hence we have the solution to the differential equation $X' = AX$ is $\vec{x}(t) = c_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} te^t \\ e^t \end{pmatrix}$.

**Importance of the matrix exponential** $e^{\mathbf{A}t}$ has columns which are the solutions to the differential equation.

## 5.2 29 Oct 2014

**Non-diagonalizable matrix as a sum of commuting matrices** Consider $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which has repeated real eigenvalues. We note that we can write $\mathbf{A}$ as a sum of matrices: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A}_1 + \mathbf{A}_2$. These matrices commute with each other (clearly, since one is the identity): $[\mathbf{A}_1, \mathbf{A}_2] = \mathbf{A}_1\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_1 = 0$. Hence when we take the exponential of the matrices, we have $e^{\mathbf{A}_1+\mathbf{A}_2} = e^{\mathbf{A}_1}e^{\mathbf{A}_2}$, which only holds when the two matrices commute. Hence we have $e^{\mathbf{A}_1 t} = e^{\mathbf{I}t} = \sum_{k=0}^{\infty} \mathbf{I}\frac{t^k}{k!} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$. Also, note that when we take $\mathbf{A}_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ hence the series only has the $k = 0$ and $k = 1$ term since all higher orders vanish: $e^{\mathbf{A}_2 t} = \mathbf{I} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t$. Hence we have $e^{\mathbf{A}t} = e^{\mathbf{A}_1 t}e^{\mathbf{A}_2 t} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$.

**Upper diagonal matrix with diagonal entries zero** Consider $\mathbf{A} = \begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then we have $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{A}^3 = \mathbf{0}$. Hence we can just calculate $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2$. In general, for such matrices, higher orders will vanish for some power, as the non-zero elements get pushed to the top right corner. Such matrices are called nilpotent if $\mathbf{A}^k = \mathbf{0}$ for some positive integer $k$.

**Equilibrium states** Consider the matrix equation $\dot{\mathbf{X}} = \mathbf{AX}$. The equilibrium states are the vectors in the kernel (null space) of $\mathbf{A}$. If the matrix $\mathbf{A}$ is invertible, the kernel is just the zero element. But nilpotent matrices are not invertible, hence we will have some equilibrium states. Note: show that nilpotent matrices are not invertible by taking the determinant of $\mathbf{A}^k = \mathbf{0}$, and noting that the determinant of matrix products is just the product of the determinants so $\det \mathbf{A} = 0$.

**General Strategy** Consider the homogenous linear first order system with constant coefficients $\dot{\mathbf{X}} = \mathbf{AX}$. Note that even if $\mathbf{A}$ is not diagonalisable, there is still a Jordan canonical form for $A$: there exists a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$, where $\mathbf{B}$ is a matrix with blocks along the diagonal.

## 5.3 30 Oct 2014 Recitation (Midterm Review)

**Useful rules for finding integrating factors for inexact equations** If:

- $\frac{M_y - N_x}{N}$ is a function of $x$, use $\mu = \exp(\int dx \frac{M_y - N_x}{N})$ as a function of $x$.
- $\frac{N_x - M_y}{M}$ is a function of $y$, use $\mu = \exp(\int dy \frac{N_x - M_y}{M})$ as a function of $y$.
- $\frac{M_y - N_x}{N - M}$ is a function of $x + y$, then use $\mu = \exp(\int d(x+y) \frac{M_y - N_x}{N - M})$ as a function of $x + y$
- $\frac{M_y - N_x}{xM - yN}$ is a function of $xy$, then use $\mu = \exp(\int d(xy) \frac{M_y - N_x}{xM - yN})$ as a function of $xy$

## 5.4 Friday 31 Oct 2014

**Matrix exponential with non-constant matrix** Set $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. Then we define $\boldsymbol{\psi} = \exp(\int_{t_0}^{t} \mathbf{A}(s)ds)$. If $\mathbf{A}(t)$ has constant coefficients, then we have that $e^{\mathbf{A}(t-t_0)} = e^{-\mathbf{A}t_0}e^{\mathbf{A}t}$. $\boldsymbol{\psi}$ is a matrix of columns that solve the differential equation. To calculate the integral, we integrate the matrix coordinate-wise.

**Inhomogeneous linear systems** We write $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{B}(t)$. Recall that in the 1D case we had $\frac{dx}{dt} = ax + b$, which had an equilibrium solution with $x = \frac{-b}{a}$. We then had the solution $x = \frac{-b}{a} + ke^{at}$, where $k$ is an arbitrary constant. If $b$ was a function of $t$ but $a$ remained a constant, then we had the solution as $x(t) = e^{at}\int_{t_0}^{t} e^{-as}b(s)ds$ using the variation of parameters by letting $k$ vary with $k$ in the constant solution. Hence by analogy, if $\mathbf{A}$ is diagonal with eigenvalues $\lambda_n$ along the diagonal, we have that $\mathbf{X} = \begin{pmatrix} e^{\lambda_1 t}\int_{t_0}^{t} e^{-\lambda_1 s}b_1(s)ds + c_1 e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t}\int_{t_0}^{t} e^{-\lambda_n s}b_n(s)ds + c_n e^{\lambda_n t} \end{pmatrix}$, where $b_n(t)$ is the $n$th element of the column vector $\mathbf{B}$. We see that the first term of each element of $\mathbf{X}$ is the solution of the non homogenous equation while the second term is the solution to the homogenous equation.

**Identifying particular solutions to inhomogeneous linear systems** Suppose we have $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + e^{rt}\mathbf{B}$, where $\mathbf{B}$ is a fixed vector not depending on $t$. The exponential suggest that we should look for a solution that contains an exponential function. We hence make the guess $\mathbf{X}(t) = e^{rt}\mathbf{V}$, where $V$ is a vector constant that we need to solve for. We plug this guess into the equation to obtain $re^{rt}\mathbf{V} = e^{rt}\mathbf{AV} + e^{rt}\mathbf{B}$, which implies that we need to match $(r\mathbf{I} - \mathbf{A})\mathbf{V} = \mathbf{B}$. To find $\mathbf{V}$, we need $r\mathbf{I} - \mathbf{A}$ to be invertible, and hence its determinant should be non-zero. This condition requires that $r$ is not an eigenvalue of $\mathbf{A}$. If $r$ is not an eigenvalue of $\mathbf{A}$, then we can pick $\mathbf{V} = (r\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ and $\mathbf{X}(t) = \mathbf{V}e^{rt}$ is a solution to the non-homogeneous equation.

**Higher Order Equations** Consider $x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}x' + a_n x = b(t)$. We can hence let $\mathbf{Y} = (y_1, y_2, \ldots, y_n)^T$ be a column vector with $y_1 = x, y_2 = \frac{dx}{dt}, \ldots, y_n = \frac{d^{n-1}x}{dt^{n-1}}$, and hence we have the equations:

$$\frac{dy_1}{dt} = y_2$$
$$\vdots$$
$$\frac{dy_n}{dt} = -a_n y_1 - a_{n-1} y_2 - \cdots - a_1 y_n$$

Hence we have the system given by $\mathbf{Y}' = \mathbf{AY}$ where the matrix $\mathbf{A}$ is given by the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix}$.

Examining the eigenvalue equation for $\mathbf{A}$, $\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ We hence see that the eigenvalues of the matrix $A$ can be determined by examining the coefficients of the original differential equation. For second order equations, this matrix $\mathbf{A}$ has the form $\begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}$. and hence $\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 + a_1\lambda + a_2 = 0$. Then the eigenvalues $\lambda_j$ with multiplicities of $m_j \geq 1$, we get solutions $e^{\lambda_j t}, te^{\lambda_j t}, \frac{t^2}{2}e^{\lambda_j t}, \ldots, \frac{t^{m_j-1}}{(m_j-1)!}e^{\lambda_j t}$. Refer to the book/notes for details about complex eigenvalues.

## 5.5 1 Nov 2014 Midterm Review

**Observations** Observe that $\frac{d}{dy}y^k = \frac{ky^k}{y}$. Hence $I' = \frac{k}{y}I \implies I = y^k$.

(Includes non-linear $p(y,t)$). Given $y' + p(y,t)y = q(t)$, solve the homogeneous equation $y' + p(y,t)y = 0$ first to obtain $y_c(t)$. Then use variation of parameters to write $y(t) = u(t)y_c(t)$ and substitute into the original equation to find conditions on $u(t)$. Solve for $u(t)$, then construct the final solution $u(t)y_c(t)$.

# Chapter 6

# Week 6

## 6.1 Monday 3 Nov 2014

**Power series solution to differential equations** Recall the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is a power series centred at $x_0$. This series converges at some point $x$ if $\lim_{N\to\infty}\sum_{n=0}^{N} a_n(x-x_0)^n$ exists. A stronger property is absolute convergence: that $\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n$ exists. An absolutely convergent series converges, but the converse is not true.

**Tests for convergence** Ratio test: if $a_n \neq 0$ and $\lim_{n\to\infty}\frac{|a_{n+1}(x-x_0)^{n+1}|}{|a_n(x-x_0)^n|} = |x-x_0|\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x-x_0|\cdot L$ exists, then the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely whenever $|x-x_0|L<1$ and diverges when $|x-x_0|L>1$. In particular, if a power series converges at $x-x_1$, then for all $x$ such that $|x-x_0|<|x_1-x_0|$, the series converges absolutely. If the series diverges at $x_1$, then it diverges for all $x$ such that $|x-x_0|>|x_1-x_0|$.

**Radius of convergence** There exists a radius of convergence $\rho \geq 0$ such that a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely when $|x-x_0|<\rho$ and diverges when $|x-x_0|>\rho$. Note that we do not know what happens at the boundary unless you know the exact values of $a_n$.

**Example of power series** E.g. Taylor series: $\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ for an infinitely differentiable function $f$. The function $f(x)$ is analytic if its Taylor series converges and the function is its Taylor series. This is a stronger condition than being infinitely differentiable.

**Example of solution by power series** Consider $y''+y=0$. We write $y(t)=\sum_{n=0}^{\infty} a_n t^n$, expanding around $t_0=0$. Taking derivatives and plugging it into the DE, we obtain:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 0$$
$$\implies \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + a_n t^n = 0$$

Fact about power series: if a power series $\sum_{n=0}^{\infty} a_n x^n$ is identically zero, then all its coefficients have to be zero. Hence we have the recursion relation: $(n+2)(n+1)a_{n+2}+a_n=0$ or $a_{n+2}=\frac{-a_n}{(n+1)(n+2)}$. We know $a_0=y(t_0), a_1=y'(t_0)$.

**Example: Airy's Equation** Consider $y''-ty=0$. Then the solution looks like $y(t)=\sum_{n=0}^{\infty} a_n t^n$, with $a_2=a_5=a_8=\ldots=0$, $a_{3n}=\frac{a_0}{2\cdot3\cdot5\cdot6\ldots(3n-1)\cdot3n}$,where we skip $4,7,10\ldots$, and $a_{3n+1}=\frac{a_1}{3\cdot4\cdot6\cdot7\ldots\cdot(3n)\cdot(3n+1)}$, where we skip $5,8,\ldots$. Note that the two free parameters are $a_0$ and $a_1$.

**Linear homogeneous 2nd order DE** Consider $P(t)y''+Q(t)y'+R(t)y=0$. Assume that $P,Q$ and $R$ are polynomials in $t$. We look for solutions that are in power series form. Assuming the $P(t)$ is non-vanishing, we can write $p(t)=\frac{Q(t)}{P(t)}, q(t)=\frac{R(t)}{P(t)}$ as rational polynomials. We know by initial conditions that $y''(t_0)=-p(t_0)y'(t_0)-q(t_0)y(t_0)$. Hence we have that $2a_2=-p(t_0)a_1-q(t_0)a_0$. Further differentiation gives $3!a_3=-2a_2p(t_0)-(p'(t_0)+q(t_0))a_1-q'(t_0)a_0$. Repeating this multiple times gives us a sequence of coefficients $a_n$ depending on $a_0$ and $a_1$ and $p^{(k)}(t_0), q^{(k)}(t_0)$. We now need to verify that the radius of convergence of this power series is positive, such that it converges at some point. Now $p$ and $q$ are infinitely differentiable at $t_0$ since they are rational polynomials. We can also write $p(t)=\sum_{n=0}^{\infty} p_n(t-t_0)^n, q(t)\sum_{n=0}^{\infty} q_n(t-t_0)^n$ since they are also analytic. We say that $t=t_0$ is an ordinary/regular point for the equation if the analytic conditions hold (p and q are their Taylor series in some neighbourhood of $t_0$). Otherwise say that $t = t_0$ is a singular point. If $t_0$ is a regular point, then there is a non-trivial radius of convergence for the solution power series, which is the minimum of the radii of convergence for the Taylor series of $p$ and $q$.

## 6.2 5 Nov 2014 Wednesday

**Singular points of differential equations** There are two kinds of singular points - regular singular equations (mild singularities) and irregular singular equations.

**Regular Singular Equations** Consider the general form: $P(t)y'' + Q(t)y' + R(t)y = 0$. Define $p(t) = \frac{Q(t)}{P(t)}, q(t) = \frac{R(t)}{P(t)}$. A point $t_0$ is regular-singular if $\lim_{t\to t_0}(t-t_0)p(t)$ is finite and $\lim_{t\to t_0}(t-t_0)^2q(t)$ is finite. For $t_0 = 0$, write $tp(t) = \sum_{n=0}^{\infty} p_n(t-t_0)^n, t^2q(t) = \sum_{n=0}^{\infty} q_n(t-t_0)^n$, where we assume that $tp(t)$ and $t^2q(t)$ are analytic (that is, $p(t)$ has a pole of order 1 at $t_0$ and $q(t)$ has a pole of order 2 at $t_0$). Assume that this holds for some positive radius of convergence $\rho > 0$. This is the case when $P, Q, R$ are polynomials. The simplest case is when $p_n = 0$ for $n \geq 1$, and $q_n = 0$ for $n \geq 1$. Hence we only have $p(t) = p_0$ and $q(t) = q_0$. Then we are looking for solutions to the differential equation $t^2y'' + p_0ty' + q_0y = 0$, which are Euler equations.

**Euler Equations** Consider $t^2y'' + p_0ty' + q_0t = 0$. Then we make the guess $y = e^{r\log t}$ since taking derivatives will give us factors of $1/t$ that will help to remove the leading $t$ coefficients of the derivatives. Substituting this into the equation, we obtain that $r$ must satisfy $t^2(r(r-1) + p_0r + q_0) = 0$, which has three cases: distinct real roots, coincident real roots and complex conjugate roots based on the values of $p_0$ and $q_0$. For real roots, the solutions are $t^{r_1}$ and $t^{r_2}$. For repeated roots, the solutions are $t^r$ and $(\log t)t^r$. For complex conjugate roots, the solutions are $t^\lambda\cos(\mu\log t)$ and $t^\lambda\sin(\mu\log t)$, where $r = \lambda \pm \mu i$.

**Regular Singular Equations Revisited** Write $t^2y'' + p(t)ty' + q(t)y = 0$. We want to look for solutions of the form $y(t) = t^r\sum_{n=0}^{\infty} a_nt^n$. Since $p$ and $q$ are analytic for this equation to be singular, we can write them using their corresponding power series $p(t) = \sum_{n=0}^{\infty} p_nt^n, q(t) = \sum_{n=0}^{\infty} q_nt^n$. Substituting this into the original equation we obtain the product of two power series (which has to be evaluated using the discrete convolution).

Recall the discrete convolution: $\sum_{n=0}^{\infty} c_nx^n\sum_{n=0}^{\infty} d_nx^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_kd_{n-k}x^n$.

Consolidating coefficients of $x^n$, we have:

$$F(r) \equiv r(r-1) + p_0r + q_0$$
$$a_0F(r)t^r + \sum_{n=1}^{\infty}\left[F(r+n)a_n + \sum_{k=0}^{n-1} a_k(r+k)p_{n-k} + q_{n-k}\right]t^{r+n} = 0$$

Now each of the coefficients have to vanish, hence we have:

$$a_0F(r) = 0$$
$$\left[F(r+n)a_n + \sum_{k=0}^{n-1} a_k(r+k)p_{n-k} + q_{n-k}\right] = 0$$

Note that this gives a recursion relation for $a_n$ in terms of the coefficients before that. But this recursion only works if $F(r+n) \neq 0$ for all $n$, otherwise we have no information regarding that particular value of $a_n$. Hence if $F(r) = r(r-1) + p_0r + q_0 = 0$ has two distinct solutions, and if we pick the larger root, then $F(r+n)$ will not vanish whenever $n \geq 1$.

**Relation between independent solutions of Euler Equation** Note that for repeated roots of the Euler equation, we have that $\log te^{r\log t} = \frac{\partial}{\partial r}e^{r\log t}$, hence one solution is the derivative of the other. Hence by analogy, we hypothesise that if we have $y_1$, we can just differentiate it to obtain the other solution.

## 6.3 Thursday 6 Nov 2014

**Exponentiating by diagonalizing** Write $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, for a known matrix $\mathbf{A}$, and the diagonalizing matrix $\mathbf{P}$ made from the eigenvectors of $\mathbf{P}$. Then $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and we can write $e^{\mathbf{A}} = e^{\mathbf{P}\mathbf{D}\mathbf{P}^{-1}} = \mathbf{P}(\sum_{n=0}^{\infty}\frac{\mathbf{D}^n}{n!})\mathbf{P}^{-1} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1}$.

## 6.4 Friday 7 Nov 2014

**Regular Singular Equations** Consider $x^2y'' + x(xp(x)) + (x^2q(x))y = 0$, where we assume that $xp(x) = \sum_{n=0}^{\infty} p_n x^n$ and $x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$ are analytic. Then we have the associated Euler equation $x^2y'' + xp_0y' + q_0y = 0$. We want to look for solutions of the form $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$. Substituting this back into the Euler equation, we obtain that $r$ must satisfy the recursion relation with $F(r) = r(r-1) + p_0 r + q_0$,

$$a_0 F(r) = 0$$
$$a_n = \frac{1}{F(r+n)} \left[ \sum_{k=0}^{n-1} a_k (r+k) p_{n-k} + q_{n-k} \right] = 0$$

We want to choose $r$ such that $F(r) \neq 0$. Note that we can write the roots of $F(r) = 0$ as $r_1$ and $r_2$. If both of these roots are real and distinct WLOG say $r_1 > r_2$, then picking $r = \max(r_1, r_2)$ will ensure that $F(r+n) \neq 0, n \geq 1$. If $r_1 = r_2$, then picking $r = r_1 = r_2$ will also ensure that $F(r+n) \neq 0$. Hence we have one solution to the differential equation with free parameter $a_0$, which is $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$. We need to find another linearly independent solution with another free parameter.

We now have several cases. If $r - 1 - r_2$ is not an integer, then $r_2 + n$ will not be a zero of $F$, since the other zero is $r_1$. Hence we can pick the other solution to have $r = r_2$ and obtain another linearly independent solution $y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n x^n$ with another free parameter $a_0$ (not the same $a_0$ as the other solution!).

If $r_1$ and $r_2$ are complex conjugates, then we can just take the real and imaginary parts of $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ to obtain two linearly independent solutions.

Now consider the case when $r_1 = r_2$, roots are real and coincident. Then we pick $y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ as one of the solutions with free parameter $a_0$. To obtain the second solution, we recall that the Euler equation had solutions $x^r$ and $x^r \log x$, which hints that the second solution can be obtained from the first by differentiating the first with respect to $r$, and evaluated at $r = r_1 = r_2$. We apply this reasoning to obtain that the second solution should be $y_2(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$. It works.

If $r_1 \neq r_2$ are real solutions of $F(r)$ with $r_1 - r_2$ is an integer, then with one solution $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$, we can construct the other solution $= a y_1(x) \log x + x^{r_2}(1 + \sum_{n=0}^{\infty} c_n(r_2) x^n)$, where $a = \lim_{r \to r_2} (r - r_2) a_n(r)$ and $c_n(r_2) = \frac{\partial}{\partial r}[(r - r_2) a_n(r)]_{r=r_2}$.

**Example** Consider $2x^2y'' - xy' + (1+x)y = 0$. Note that $x = 0$ is the regular singular point. Then we write $xp(x) = -\frac{1}{2} = p_0$ and $x^2q(x) = \frac{1+x}{2}$ and $q_0 = \frac{1}{2}, q_1 = \frac{1}{2}$. The associated Euler equation is $2x^2y'' - xy' + y = 0$. We want to look for solutions in the form $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$. Plugging this guess into the original equation (not associated Euler equation), we obtain $\sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1}$. Combining terms, we obtain:

$$a_0(2r(r-1) - (r+1))x^r + \sum_{n=1}^{\infty} \left[ (2(r+n)(r+n+1) - (r+n) + 1)a_n + a_{n-1} \right] x^{r+n} = 0$$

The zero-th order term gives the two exponents for the Euler equation: $(2r(r-1) - (r+1)) = (r-1)(2r-1) \implies r_1 = 1, r_2 = \frac{1}{2}$. In the recursion equation,

$$(2(r+n)(r+n+1) - (r+n) + 1)a_n + a_{n-1} = 0$$
$$\implies a_n = \frac{-a_{n-1}}{((r+n)-1)(2(r+n)-1)}, n \geq 1$$

Hence if we pick $r = r_1 = 1$, the denominator does not vanish.

Consider the first solution, with $r = r_1 = 1$. Then we have that $a_n = \frac{-a_{n-1}}{(2n+1)n}$. This will become $a_n = \frac{(-1)^n a_0 2^n}{(2n+1)!}$. The general solution can be written as $y_1(x) = a_0 x \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n \right)$, which has a radius of convergence infinity. The radius of convergence can be calculated using the ratio gets $\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \frac{2|x|}{(2n+2)(2n+3)} = 0$. For the second solution, we let $r = \frac{1}{2}$. Hence we have that $a_n = \frac{(-1)^n 2^n a_0}{(2n)!}$ and general solution $y_2(x) = a_0 x^{1/2} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n \right)$.

# Chapter 7

# Week 7

## 7.1 Monday 10 Nov 2014

**Integral transforms** Consider the map $f(t) \mapsto F(f)(s) = \int_{\alpha}^{\beta} K(s,t)f(t)dt$, where $K(s,t)$ is called the integral kernel of $F$. Note that $\alpha$ and $\beta$ can be any two complex numbers (including $\pm\infty$).

**Laplace Transform** Define $\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st}f(t)dt$. We want the integral to be convergent. Hence we want to find $f(t)$ such that $|f(t)| \leq Ke^{at}$ for some $K \geq 0, a > 0$ and $t \geq T$ sufficiently large. This just says that we want $f(t)$ to grow slower (or as fast as) an exponential. In such a case, $(f)(s)$ will be defined for at least $s > a$. This is because we can write the integral as $\int_0^{\infty} e^{-st}f(t)dt = \int_0^{T} e^{-st}f(t)dt + \int_T^{\infty} e^{-st}f(t)dt$, where $T$ is finite. Note that the second term is less or equal to $\int_T^{\infty} e^{-st}Ke^{at}dt = \int_T^{\infty} Ke^{(a-s)t}dt$, which is convergence when $s > a$.

**Example 1** Consider the function $f(t) = 1$. Then its LT is just $\mathcal{L}(1)(s) = \frac{1}{s}$, which is defined for $s > 0$.

**Example 2** Consider the exponential $f(t) = e^{at}, t \geq 0$. Then we have $\mathcal{L}(e^{at})(s) = \frac{1}{s-a}$, defined for $s > a$.

**Example 3** Consider $f(t)\sin(at), t \geq 0$. Then the Laplace transform is $\mathcal{L}(\sin(at))(s) = \int_0^{\infty} e^{-st}\sin(at)dt = \frac{a}{s^2+a^2}$ after integrating by parts twice.

**Example 4** Note that the Laplace transform of $t$ is $\mathcal{L}(t) = \frac{1}{s^2}$.

**Properties of Laplace Transforms**

- **Linearity**: $\mathcal{L}(\alpha f(t) + \beta g(t))(s) = \alpha\mathcal{L}(f(t))(s) + \beta\mathcal{L}(g(t))(s)$.
- **Laplace Transform of a Derivative**: Consider a continuous and differentiable function $f$, with a first derivative that is piecewise continuous. Suppose further that it is bounded by $|f(t)| \leq Ke^{at}$ and $|f'(t)| \leq K_1e^{a_1t}$ as well. Then we have that $\mathcal{L}(f'(t))(s) = s\mathcal{L}(f(t))(s) - f(0)$. Hence the Laplace transform of a derivative just becomes a multiplication by $s$ (and subtracting a constant). **Proof**: Let $t_1, \ldots, t_n$ be n discontinuities of $f'$. Then we can split the integral $\int_0^{\alpha} e^{-st}f'(t)dt$, where $\alpha > \max(t_i)$, into $\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} e^{-st}f'(t)dt$, where $k_0 = 0$. Perform integration by parts on each term separately (then combining terms etc) to obtain that the sum is equal to $e^{-s\alpha}f(\alpha) - e^0f(0) + s\int_0^{\alpha} e^{-st}f(t)dt$. Taking the limit as $\alpha \to \infty$, we obtain that $\int_0^{\infty} e^{-st}f'(t)dt = s\int_0^{\infty} e^{-st}f(t)dt - f(0)$ as required.
- **Laplace Transform of Higher Derivatives**: Note that $\mathcal{L}(f''(t)) = s^2\mathcal{L}(f) - sf(0) - f'(0)$. In general,

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0) = s^n\mathcal{L}(f) - \sum_{k=1}^{n} s^{k-1}f^{(n-k)}(0)$$

- **Laplace transform of differential equations** We can take the Laplace transform of an entire differential equation since the Laplace transform is linear. We hence obtain an algebraic equation for the Laplace transform of $y$ in terms of $s$ and the initial conditions. Solve for $\mathcal{L}(y)$ and take its inverse Laplace transform to find $y$.

**Step Functions (Heaviside Function)** Write $u_c(t) = \begin{cases} 0, t < c \\ 1, t \geq c \end{cases}$. The Laplace transform of this function is clearly $\int_0^{\infty} u_c(t)e^{-st}dt = \int_c^{\infty} e^{-st}dt = \frac{1}{s}e^{-sc}$, defined for $s > 0$.

**Step function multiplied by another function** Consider $g(t) = f(t)u(t-c)$. Then the Laplace transform of $g(t)$ is $\mathcal{L}(g) = \int_0^\infty f(t)u(t-c)e^{-st}dt = \int_c^\infty f(t)e^{-st}dt = \int_0^\infty e^{-(\zeta+c)s}f(\zeta)d\zeta = e^{-cs}\mathcal{L}(f)$, where we have made the change of variable $\zeta = t - c$.

## 7.2 12 Nov 2014

**Useful Laplace Transforms** $\mathcal{L}(t^r) = \frac{\Gamma(r+1)}{s^{r+1}} = \frac{r!}{s^{r+1}}$.

**Gamma Function** $\Gamma(r+1) = \int_0^\infty e^{-u}u^r du$. There is a recursion relation $\Gamma(r+1) = r\Gamma(r)$, $\Gamma(1) = 1$, so $\Gamma(n+1) = n!$.

**Multiplication by exponential** Consider $e^{at}f(t)$. Then the Laplace transform is $\mathcal{L}(f)(s-a), s > a$.

**Dirac Delta Functions** Defined to be zero for all $t \neq 0$ and infinity at $t = 0$ such that $\int_{-\infty}^\infty \delta_\tau(t)dt = 1$. Note that the Dirac delta function is not a function but it is a distribution. A distribution $\Lambda$ is a continuous linear transformation that takes $C^\infty(\mathbb{R})$ smooth functions that are zero outside some compact set (i.e. has compact support) as its input and maps them to the reals.

**Example of a distribution** Consider a function $h : \mathbb{R} \to \mathbb{R}$ that is piecewise continuous. Then define the distribution $\Lambda(h) = \int_{-\infty}^\infty h(t)f(t)dt$.

**Delta function as a distribution** Define a distribution acting on functions as $\Lambda_{\delta\tau}(f) = \int_{-\infty}^\infty \delta_\tau(t)f(t)dt = \frac{1}{2\tau}\int_{-\tau}^\tau f(t)dt$. Hence we define the functional $\delta : C^\infty(\mathbb{R}) \to \mathbb{R}, \delta(f) = \lim_{\tau\to 0} \Lambda_{\delta\tau} = f(0)$.

**Laplace Transform of a Distribution** Define $\mathcal{L}(\Lambda(f)) = \Lambda(\mathcal{L}(f))$. The Laplace transform of a distribution acting on a function is the distribution acting on the Laplace transform of the function.

**Derivative of a distribution** $\Lambda'(f) = -\Lambda(f')$, where the minus sign is related to the minus sign in integration by parts.

## 7.3 13 Nov 2014 Thursday

**Useful Identities**

- $\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}(f(t)))$.
- $\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))$

**Exponential Order** A function $f(t)$ is of exponential order $\alpha$ if there exists $T, M$ such that $|f(t)| \leq Me^{\alpha t}, \forall t \geq T$. Alternatively, say that $\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}}$ is finite.

## 7.4 14 Nov 2014 Friday

**Review** Recall that the delta function can be thought of as a limiting distribution $\lim_{\tau\to 0} \Lambda_{d,\tau}$, where $\Lambda_{d,\tau}$ refers to the distribution acting on smooth functions with compact support (i.e. $C_c^\infty(\mathbb{R})$ functions) $\Lambda_{d,\tau}(f) = \frac{1}{2\tau}\int_{-\tau}^\tau f(t)dt$. Then $\delta(t) = \lim_{\tau\to 0} \Lambda_{d,\tau}(f) = f(t)$.

**Laplace Transform of a Distribution** Consider $\Lambda_h(f) = \int h(t)f(t)dt$. Then $\Lambda_h(\mathcal{L}(f)) = \int h(s)\mathcal{L}(f)(s)ds$. Writing this out explicitly, we have that this is equal to $\int h(s)\left[\int_0^\infty e^{-st}f(t)dt\right]ds$. Since all these integrals are assumed to converge, we can exchange the order of integration to obtain: $\int_0^\infty f(t)\left[\int_0^\infty e^{-st}h(s)ds\right]dt$. But the integral inside can be viewed as the Laplace transform of h in terms of t Hence we have that it is equal to $\int \mathcal{L}(h)(t)f(t)dt = \Lambda_{\mathcal{L}(h)}(f)$.

**Laplace Transform of the Delta Function** Consider $\mathcal{L}(\delta_{t_0})(f) = \delta_{t_0}(\mathcal{L}(f))$ as per the proof immediately above. The RHS is equal to $\mathcal{L}(f)(t_0) = \int_0^\infty e^{-t_0 s}f(s)ds$ as per the definition of the delta function distribution. Hence (somehow) we can think of $\mathcal{L}(\delta_{t_0}) = e^{-t_0 s}$.

**Derivative of a distribution** Consider we a differentiable function $h$. We can interpret the function as a distribution by defining $\Lambda_h(f) = \int_\mathbb{R} h(t)f(t)dt$. We want the derivative of the distribution to equal the distribution associated with the derivative of the function $\Lambda_{h'}(f) = \int_\mathbb{R} h'(t)f(t)dt = [h(t)f(t)]_{-\infty}^\infty - \int_\mathbb{R} h(t)f'(t)dt$ by integration by parts. Now the first term vanishes because $f(t)$ has compact support and hence vanishes outside the supported region. Note that the second term hence becomes $-\Lambda_h(f')$. Hence if $h$ is differentiable, we define the derivative of the distribution to be $\Lambda_{h'}(f) = -\Lambda_h(f')$.

Hence the derivative of any distribution $\Lambda'(f) = -\Lambda(f')$.

**Derivative of the Delta Function Distribution** We use the definition of the derivative for distributions to state that $\delta'_{t_0}(f) = -\delta_{t_0}(f') = -f'(t_0)$.

**Heaviside theta distribution** Consider the step function $u_c$ with jump at $t = c$. The distribution associated with this function is $\Lambda_{u_c}(f) = \int_{\mathbb{R}} u_c(t)f(t)dt = \int_c^\infty f(t)dt$. The distributional derivative of this is $\Lambda'(u_c) = -\Lambda(u'_c) = -\int_c^\infty f'(t)dt = -[f(t)]_c^\infty = -f(c) = \delta_c(f)$ because $t = \infty$ is outside the compact support. Hence $\Lambda'_{u_c} = \delta_c$.

**Example with delta function** Consider $2y'' + y' + 2y = \delta_{t_0=5}$ with initial condition $y(0) = 0, y'(0) = 0$. Now the RHS is a distribution, so we need the LHS to be a distribution as well. We take the Laplace transform of the LHS to get $(2s^2 + s + 2)Y(s)$. On the RHS we have $e^{-5s}$. Hence we have that $Y(s) = \frac{e^{-5s}}{2s^2+s+2}$

# Chapter 8

# Week 8

## 8.1 Monday 17 Nov 2014

**Laplace Transform of Convolution is product of Laplace transforms** Define the convolution $f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\tau)f_2(\tau)d\tau$. Then the Laplace transform of $f$ satisfies:

$$\mathcal{L}(f) = \mathcal{L}(f_1)\mathcal{L}(f_2)$$

is just the product of the individual Laplace transform.

**Proof** Consider $\mathcal{L}(f) = \int_0^\infty e^{-st} \int_0^t f_1(t-\tau)f_2(\tau)d\tau dt$ and compare it with $\left(\int_0^\infty e^{-st} f_1(t)dt\right)\left(\int_0^\infty e^{-st'} f_2(t')dt'\right)$. Then we have that the product is $\int_0^\infty \int_0^\infty e^{-s(t+t')} f_1(t)f_2(t')dtdt'$. Note that $0 \le t \le T$ so $0 \le t+t' \le T+t'$. Make a change of variable $t+t' = u \implies t = u-t'$. Rewriting the integral in terms of the new variable, we have: $\int_0^\infty \int_0^u e^{-su} f_1(u-t')f_2(t')dudt' = \int_0^\infty e^{-su} \int_0^u f_1(u-t')f_2(t')dt'du = \mathcal{L}(f)$.

**Properties of Convolution**

- Commutative $f * g = g * f$
- Distributive over sums $f * (g_1 + g_2) = f * g_1 + f * g_2$
- Associative $f * (g * h) = (f * g) * h$

**General Case of Non-Homogeneous Linear Constant Coefficient Second Order DE** Consider $ay'' + by' + cy = g(t)$ with initial conditions $y(0) = y_0, y'(0) = y'_0$. Take the Laplace Transform of the LHS to obtain:

$$(as^2 + bs + c)Y(x) - (as + b)y_0 - ay'_0 = G(s)$$

where $\mathcal{L}(y) = Y(s)$ and $\mathcal{L}(g) = G(s)$. Introduce the function $\Phi(s) = \frac{(as+b)y_0 + ay'_0}{as^2+bs+c}$ which contains all the information about the initial conditions. Also introduce $\Psi(s) = \frac{G(s)}{as^2+bs+c}$ which has information about the non-homogeneous term. Then:

$$Y(s) = \Phi(s) + \Psi(s)$$

Hence we want to find $y(t)$ such that $\mathcal{L}(y) = Y(s)$. Since the Laplace transform is linear, we can write $y$ as the sum of two terms $y = \phi(t) + \psi(t)$ such that $\mathcal{L}(\phi) = \Phi(s)$ and $\mathcal{L}(\psi) = \Psi(s)$. Hence we want to find the Inverse Laplace Transforms of the two pieces of $Y(s)$, then add them together to obtain the two pieces of $y(t)$. Observe that if $g(t) = 0$, then $\Psi(s) = 0 \implies Y(s) = \Phi(s)$, so $\phi(t)$ is the solution of the homogeneous equation given the **same initial conditions**. Observe further that if $y_0 = y'_0 = 0$, then $\Phi(s) = 0$ and $Y(s) = \Psi(s)$ so $\psi(t)$ is the solution to the non-homogeneous differential equation with zeros as the initial conditions.

Hence we find the solution by doing three things:

1. Solve the homogeneous case with the same initial conditions,
2. Solve the inhomogeneous case with initial conditions being zero.

3. Add the two contributions together to obtain the final solution.

Define the function $H(s) = \frac{1}{as^2+bs+c}$ so that $\Psi(s) = H(s)G(s)$. $H(s)$ contains information about the original differential operator in $ay'' + by' + cy$. $H(s)$ is known as the transfer function or propagator. Note that $H(s)$ contains no information about the initial conditions of the system or the inhomogeneous part. Note further that $H(s)$ is the solution to the Laplace transform of the equation:

$$ay'' + by' + cy = \delta(t)$$

with initial conditions $y(0) = 0, y'(0) = 0$. Note that $\delta(t)$ is the delta function centred at zero. In light of this, $H(s)$ is also called the Fundamental Solution or Green Function.

Now to find $\psi(t)$ we want to invert $\Psi(s)$, so:

$$\begin{aligned}\psi(t) &= \mathcal{L}^{-1}(H(s)G(s)) \\ &= \int_0^t h(t-\tau)g(\tau)d\tau\end{aligned}$$

in view that the Laplace transform of a convolution is the product of the individual Laplace Transforms. Hence we want to find $h(t)$, the inverse Laplace transform of $H(s)$.

Putting it all together, we can write:

$$y(t) = \phi(t) + \int_0^t h(t-\tau)g(\tau)d\tau$$

where we have the following components: $\phi(t)$ the solution to the homogeneous equation, $h(t)$ the inverse Laplace transform of the Fundamental solution and $g(t)$ the non-homogeneous term.

**Solving Difference Equations with Discretized Time** Recall that we have always treated $y(t)$ as a function of a continuous time variable. In this case, we want to replace time with a set of discrete values $t_n$ so that $y_n = y(t_n)$. Then the equations we wish to solve are of the form:

$$\begin{aligned}y_{n+1} &= F(n, y_n) \\ y_0 &= y(t_0), \text{ the initial condition}\end{aligned}$$

The equivalent of autonomous equations in this case is that the function $F$ is just a function of $y_n$ and not on $n$.

$$y_{n+1} = F(y_n), \quad \text{autonomous case}$$

Note that in general for difference equations $y : \mathbb{N} \to \mathbb{R}$ and $F : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. Given $y_0 = \alpha$, define $y_k = F^k(\alpha)$ to be the orbit of $\alpha$ under iterations of the function $F$.

**Example** Consider $y_{n+1} = \rho y_n, y_0 = \alpha$, so if $0 < |\rho| < 1$, then $y_n = \rho^n \alpha \to 0$ as $n \to \infty$. If $\rho = 1$, then $y_n = \alpha, \forall n$, If $\rho > 1$, then $y_n$ diverges. If $\rho = -1$, then $y_n$ does not have a limit because it keeps oscillating between $\alpha$ and $-\alpha$.

**Example** Consider $y_{n+1} = \rho y_n + b_n, y_0 = \alpha$. Then $y_1 = \rho\alpha + b_0$, etc. In general $y_n = \rho^n \alpha + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j$. If $b_n = b, \forall n, \rho \neq 1$ then $y_n = \rho^n(\alpha - \frac{b}{1-\rho}) + \frac{b}{1-\rho}$.

## 8.2 Own Notes on Distributions (Distributions, Complex Variables, and Fourier Transforms by Bremermann)

**Functional** Let $\{f_n\}$ be a sequence of functions such that the integrals $\int_{-\infty}^{\infty} f_n(t)\phi(t)dt$ exist for all $n$ and for all $\phi$ from a given class of function. Then define $< f_n, \phi >= \int_{-\infty}^{\infty} f_n(t)\phi(t)dt$ and $< f, \phi >= \lim_{n\to\infty} < f_n, \phi >$ be called a functional. $\phi$ is called a test function.

**Schwartz Distribution** A Schwartz distribution has a class of test functions $\phi$ that are smooth with compact support.

**Delta function as a distribution** $< \delta, \phi >= \phi(0)$ for all complex valued functions $\phi$ defined on the real axis.

**Support** The support of a function is the closure of the set of all points for which the function is non-zero.

## 8.3 Wednesday 19 Nov 2014

**Example** Consider $\frac{dy}{dt} = ry(1-\frac{y}{T})$, which has the continuous form. If we discretize this, we replace $\frac{dy}{dt}$ with a finite increment $\frac{y_{n+1}-y_n}{h}$, where $h$ is the length of time between consecutive $y_j$. Then we have:

$$\begin{aligned} \frac{y_{n+1}-y_n}{h} &= ry_n(1-\frac{y_n}{T}) \\ \implies y_{n+1}-y_n &= hry_n - hr\frac{y_n^2}{T} \\ \implies y_{n+1} &= (hr+1)y_n - hr\frac{y_n^2}{T} \\ \implies y_{n+1} &= \rho y_n(1-\frac{y_n}{K}), \rho = (1+hr), K = \frac{T}{hr}(1+hr) \end{aligned}$$

Make the substitution $u_n = \frac{y_n}{K}$ so we have the difference equation:

$$u_{n+1} = \rho u_n(1-u_n)$$

Examine the equilibrium solutions: $u_n = 0, \forall n$, $u_n = \frac{\rho-1}{\rho}, \forall n$. We now examine the behaviour of the other solutions. Define $f(x) = \rho x(1-x)$. Then the fixed point occurs when $x = 0, x = \frac{\rho-1}{\rho}$. Also, the roots of the function are at $x = 0$, $x = 1$. Consider the case when $\rho - 1 < 0$. If we pick the initial condition $0 < x_0 < 1$, then we observe that the orbit $f^k(x_0) = x_k$ converges to the fixed point $x = 0$ when $k \to \infty$. $x = 0$ is a stable fixed point in this case.

Now consider the case when $\rho > 1$. Then the fixed points are at the origin and at the positive position $x = \frac{\rho-1}{\rho} < 1$. Then if we pick $0 < x_0 < 1$, then the orbit $f^k(x_0) = x_k$ will converge to the fixed point $x = \frac{\rho-1}{\rho}$. Observe that $x = 0$ is now an unstable fixed point, while $x = \frac{\rho-1}{\rho}$ is a stable fixed point.

Note that when $\rho$ is sufficiently large, then there is the possibility that we can have a periodic orbit that does not go closer to the fixed point.

**Review: Systems of linear equations** Recall that in $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is solved by examining the eigenvalues and eigenvectors of $\mathbf{A}$.

**Notes on sketching phase diagrams**

- Sketch in the lines representing the eigenvectors into the phase plane.
- If the eigenvalue corresponding to an eigenvector is negative, the direction of motion along that eigenvector will be towards the origin.
- Examine the relative magnitude of the eigenvalues. If the solution varies more rapidly along one eigenvector direction (i.e. eigenvalue is more negative) then the curve will "stick" to the slower axis.

**Complex Eigenvalues** Take the real and imaginary part of $\vec{v}e^{(\mu+\lambda i)t}$, which are $(e^{\mu t}\cos\lambda t, -e^{\mu t}\sin\lambda t)$ and $(e^{\mu t}\sin\lambda t, e^{\mu t}\cos\lambda t)$. This will trace out a circle in the phase space, but since there is an exponential factor, it will become a spiral that goes to the origin if the real part of the eigenvalue is negative and to infinity of the real part is positive.

**Repeated eigenvalues (i.e. non-diagonalizable)** With repeated eigenvalues, there will not be a second linearly independent eigenvector.

## 8.4 Friday 21 Nov 2014

**Repeated eigenvalues (non-diagonalizable)** We will be able to obtain one repeated eigenvalue $\lambda$ and one associated eigenvector $\vec{\zeta}$. We know that one solution will be of the form $\vec{x}_1 = \vec{\zeta}e^{\lambda t}$. We want to find a second solution of the form $\vec{x}_2 = \vec{\zeta}te^{\lambda t} + \vec{\eta}e^{\lambda t}$ for some vector $\vec{\eta}$. Substituting this guess into the original differential equation gives $(\mathbf{A} - \lambda\mathbf{I})\vec{\eta} = \vec{\zeta}$. Now we note that $\vec{\zeta}$ is in the kernel of $(\mathbf{A} - \lambda\mathbf{I})$ because it is an eigenvector of $\mathbf{A}$. Hence we can find the solution with $\vec{\eta} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\vec{\zeta}$. Take the linear combination of $\vec{x}_1$ and $\vec{x}_2$ to find the general solution.

**General solution to a linear matrix DE with constant coefficients** Write $\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t)$. There are several different possibilities based on the eigenvalues of $\mathbf{A}$ :

- Eigenvalues are real, distinct, and positive. The origin is an unstable node.
- Eigenvalues are real, distinct, and negative. The origin is a stable node.
- Eigenvalues are real, distinct, and have different signs. The origin is a saddle/non-stable node.
- Eigenvalues are complex conjugates. Trajectories are spirals. Origin is stable if the real part is negative, unstable if the real part is positive. If the real part is zero, then we get circles or ellipses (called centres).
- Eigenvalues are real and repeated. Origin is stable or unstable depending on the sign of the eigenvalue. Trajectories follow two regimes of flow: for small $t$ the $e^{\lambda t}$ term dominates, and for large $t$ the $te^{\lambda t}$ term dominates.

**Alternative way to describe behaviour** Write the characteristic polynomial as $\det(A) + tr(A)\lambda + \lambda^2$. The behaviour depends on the discriminant of the polynomial $tr(A)^2 - 4\det(A)$. For positive discriminant, we have real and distinct eigenvalues, for negative discriminants, we get complex eigenvalues and for zero discriminant, we have real repeated eigenvalues. Also, we can determine the signs of the eigenvalues by looking at the determinant of the matrix, which is the product of the eigenvalues. If the determinant has negative sign, the eigenvalues have opposite sign, and if the determinant is positive the eigenvalues have the same sign. To see the exact sign of the eigenvalues in the latter case, look at the trace, which is the sum of the eigenvalues.

**Graphical depiction** Put $\det(A)$ on the y-axis and $tr(A)$ on the x-axis. Plot the parabola $\det(A) = \frac{1}{4}tr(A)^2$. Along the vertical axis, the trace is zero, which corresponds to pure imaginary eigenvalues and hence centers. Above the parabola, we have that the discriminant is negative and hence we have complex conjugate eigenvalues. Left of the vertical axis, we have negative trace and hence the spirals have negative real part eigenvalues and hence are stable. Right of the vertical axis, we have positive trace and hence positive eigenvalue real parts and hence unstable spirals. Below the parabola, there are 4 different cases depending on the sign of $tr(A)$ and $\det(A)$. Above the horizontal axis, the determinant is positive and hence we have either stable or unusable nodes. Below the horizontal axis, we have saddles because the determinant is negative and hence the eigenvalues have opposite signs.

**Behaviour of non-linear systems** Examine the behaviour in a small neighbourhood of a point by taking first order linear approximations.

**Example: Competing species dynamics** Construct the non-linear system:

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$\frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x)$$

Note that without the third term, the differential equations are logistic in nature. We first look for equilibrium points $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$. We hence want to solve:

$$x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0$$
$$y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0$$

Observe that $x = 0$ solves the first equation. If $x = 0$, then $y = 0$ or $y = \frac{\epsilon_2}{\sigma_2}$ solves the second equation. Alternatively, if $y = 0$, then the first equation will be solved if $x = 0$ or $x = \frac{\epsilon_1}{\sigma_1}$. We now want to check if there are solutions where neither $x$ nor $y$ are zero. This is equivalent to solving the linear system:

$$\sigma_1 x + \alpha_1 y = \epsilon_1$$
$$\sigma_2 y + \alpha_2 x = \epsilon_2$$
$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma_1 & \alpha_1 \\ \alpha_2 & \sigma_2 \end{pmatrix}^{-1} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

Hence we have the four equilibrium points: $(0,0), (0, \frac{\epsilon_2}{\sigma_2}), (\frac{\epsilon_1}{\sigma_1}, 0)$ and the one from the linear system solution above. We can examine the behaviour of the system around each of the equilibrium points.

Consider the point $(0,0)$. We linearize the system by dropping all the non-linear terms (in first approximation, since $y^2, xy, x^2$ are small near the origin):

$$\frac{dx}{dt} = \epsilon_1 x$$
$$\frac{dy}{dt} = \epsilon_2 y$$

This clearly has solutions:

$$(x, y) = (1, 0)e^{\epsilon_1 t} + (0, 1)e^{\epsilon_2 t}$$

hence we observe that $(0,0)$ is an unstable equilibrium point. The system behaves as if the two species are independently growing without considering interactions between the two populations.

# Chapter 9

# Week 9

## 9.1 24 Nov 2014 Monday

**Continuation of non-linear growth models** Recall the model:

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$\frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x)$$

which has 4 equilibrium points.

**Linearization at equilibrium points** Assume we have an equilibrium point at $(x_e, y_e)$. Define the new variables $u = x - x_e, v = y - y_e$, and then drop all non-linear terms in $u$ and $v$ that are small near the equilibrium point.

**The next equilibrium point** Now consider $(x_e, y_e) = (0, \frac{\epsilon_2}{\sigma_2})$ so we define the new variables $u = x, v = y - \frac{\epsilon_2}{\sigma_2}$. Then we write:

$$\frac{du}{dt} = \epsilon_1 u - \frac{\epsilon_2}{\sigma_2} u$$
$$\frac{dv}{dt} = \frac{\epsilon_2}{\sigma_2}(-v\sigma_2 - \alpha_2 u)$$

Hence examine the coefficient matrix $A$ and determine its behaviour based on the values of the determinant and trace.

**General description of method**

- Identify equilibrium points
- Linearize coupled equations at each equilibrium point
- Identify the determinant and trace of each coefficient matrix at each equilibrium point to identify type of trajectory near the point.
- Calculate eigenvectors and eigenvalues at each point to sketch in the phase portrait near each point.
- Attempt to connect the phase portraits over the whole plane by connecting arrows

**Prey-Predator model (Lotka/Volterra)** Consider:

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy$$

with $a, c, \alpha, \gamma > 0$. Note that in the absence of the other species, the $x$ population will grow exponentially while the $y$ population will die off exponentially. Hence $x$ is the prey while $y$ is the predator. Then the cross-terms represent the effect of encounters between prey and predator, which is bad for the prey but good for the predator.

We want to find the equilibrium solutions and linearize it near each point.

One trivial point is the origin, which is clearly a saddle point since the coefficients of x and y have opposite sign.

Now consider the other equilibrium points. We solve the simultaneous equations to obtain that the other equilibrium point is $x = \frac{c}{\gamma}, y = \frac{a}{\alpha}$. Then around this equilibrium point, we have:

$$u \equiv x - \frac{c}{\gamma}$$
$$v \equiv y - \frac{a}{\alpha}$$
$$\frac{du}{dt} = \frac{-c\alpha}{\gamma}v$$
$$\frac{dv}{dt} = \frac{a\gamma}{\alpha}u$$

Observe that the coefficient matrix has zero trace and positive determinant, hence the equilibrium point will be a centre. Hence the trajectories circle around the equilibrium point in stable fashion, and neither population goes extinct around this equilibrium point.

**Example: Damped pendulum** Consider the equations of motion:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

Note that the equilibrium solutions are $y = 0$ and $\sin x = 0$, hence we write $x = \pm n\pi, n \in \mathbb{N}$. Perform a linearisation near $(2n\pi, 0), n \in \mathbb{Z}$. Then we have:

$$\frac{du}{dt} = v$$
$$\frac{dv}{dt} = -\omega^2 u - \gamma v$$

where we have replaced $\sin u \approx u$ for small u. The trace of the coefficient matrix is $-\gamma$ and the determinant is $\omega^2$. If $\gamma = 0$, then we have no damping and we will have centers at this equilibrium point. If $\gamma > 0$, then we will have stable spirals instead.

## 9.2 Wednesday 26 Nov 2014

**Damped pendulum** Recall the equations of motion of the damped pendulum:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

which has equilibrium solutions when $y = 0, \sin x = 0 \implies x = m\pi, m \in \mathbb{Z}$. We can consider the cases when $x = 2n\pi$ and when $x = (2n+1)\pi$, the even and odd multiples of $\pi$. Linearizing around $(2n\pi, 0)$, we have the matrix equation:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Hence we have the following cases:

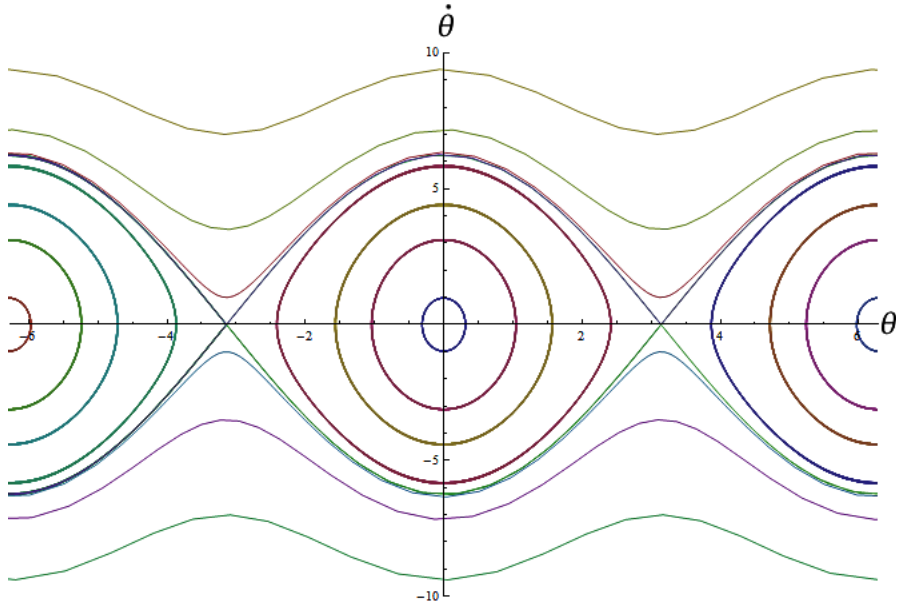
- If $\gamma = 0$, then we have the trace is zero and the determinant is positive, and hence we get centers.

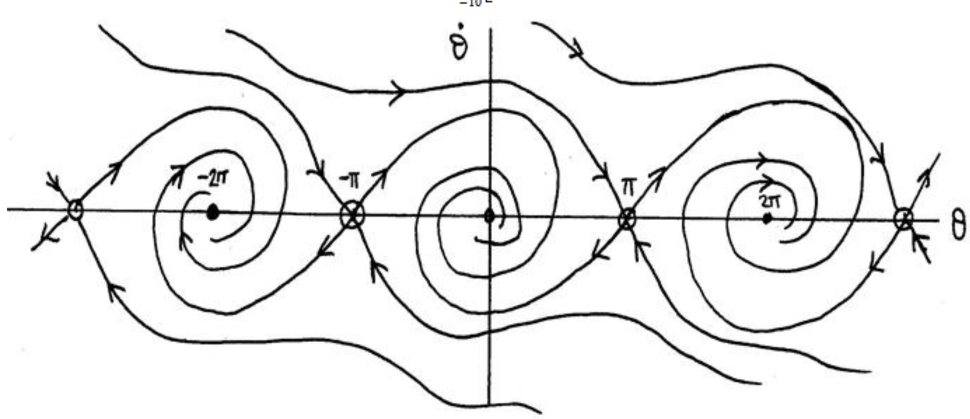
- If $\gamma > 0$ then we get that the trace is negative and the determinant is positive and hence we get stable spirals.

Now consider the linearisation around $((2n+1)\pi, 0)$. Then we make the approximation $\sin((2n+1)\pi + u) \approx -\sin u$. Then we get the matrix equation:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Then we have that the determinant is negative. Hence all the solutions around the equilibrium point are saddles. The phase diagrams are as such (for frictionless and with friction):




**Energy** Note that we can write the energy of the undamped pendulum as:

$$V = (1 - \cos x) + \frac{1}{2\omega^2}y^2$$

Examining the time derivative:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$
$$= y\sin x + \frac{y}{\omega^2}(-\omega^2 \sin x) = 0$$

hence the energy stays the same along the solution, and the solutions only move along level sets of $V$.

**Energy of the damped pendulum** We write the energy now with $\gamma > 0$:

$$\frac{dV}{dt} = y\sin x + \frac{y}{\omega^2}(-\omega^2 \sin x - \gamma y) = \frac{-\gamma}{\omega^2} y^2$$

Now note that $V$ is not constant along trajectories, but instead will be decreasing for all positions with $y \neq 0$. Hence the trajectories will be cutting through lower and lower level sets of $V$.

**Periodic orbits** Consider the non-linear system:

$$\frac{dx}{dt} = x + y - x(x^2 + y^2)$$
$$\frac{dy}{dt} = -x + y - y(x^2 + y^2)$$

First look for equilibrium solutions. Substituting $x^2 + y^2$ from one equation to another, we require that $x^2 + y^2 = 0$, which only occurs when $x = y = 0$. The linearization at $(0, 0)$ is just:

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = -x + y$$

and we compute the trace to be 2 and the determinant to be 2, and hence it is an unstable spiral.

However, further away from the origin, the linearisation will not be valid. The presence of $x^2 + y^2$ indicates that we should look at the equation in polar coordinates. Define $r^2 = x^2 + y^2 \implies r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}, x = r\cos\theta, y = r\sin\theta$.
Observe that:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = x(x + y) + x^2(x^2 + y^2) + y(-x + y) - y^2(x^2 + y^2) = x^2 + y^2 - (x^2 + y^2)^2 = r^2 - r^4$$

Hence we have the polar coordinate equation:

$$r\frac{dr}{dt} = r^2 - r^4 = r^2(1 - r^2)$$

Now consider the angular equation. Now take the difference between the original equations to obtain:

$$y\frac{dx}{dt} - x\frac{dy}{dt} = y(x + y) - xy(x^2 + y^2) - x(-x + y) + xy(x^2 + y^2) = x^2 + y^2 = r^2$$

This expression is useful since:

$$y\frac{dx}{dt} - x\frac{dy}{dt} = -r^2\frac{d\theta}{dt}$$

Hence we have the theta dependence:

$$\frac{d\theta}{dt} = -1$$

or that the angular velocity is constant on the solution.

The radial equation has two critical points at $r = 0$ and $r = 1$. The second critical point was missed in the Cartesian coordinates. This is because traversing a circle of radius 1 at constant speed corresponds to a critical point in polar coordinates but is not stationary in Cartesian coordinates.

Note that the presence of the unit circle as a solution means that there are two regimes of solutions: inside the unit circle and outside the unit circle, and the solutions do not cross the unit circle due to uniqueness.

We can solve the radial equation by separation of variables:

$$\begin{aligned}\frac{dr}{dt} = r(1-r^2) \implies \frac{dr}{r(1-r^2)} = dt\\ \implies dr\left(\frac{1}{r} + \frac{r}{1-r^2}\right) = dt\\ \implies \log(r|1-r^2|^{-1/2}) = t - t_0\\ \implies \frac{r^2}{1-r^2} = \pm e^{2(t-t_0)}\\ \implies r^2 = \frac{1}{Ce^{-2t}+1}, C = e^{2t_0}\end{aligned}$$

observe that for all trajectories, the presence of the decaying exponential indicates that all the trajectories are limiting to a circle $r = 1$.

# Chapter 10

# Week 10

## 10.1 Monday 1 Dec 2014

**Closed orbits in Non-linear systems: Criterion 1** Given:

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

where $F$ and $G$ have continuous partial derivatives in $x$ and $y$ in some region of the plane. If there is a limiting cycle in this region D, then it must enclose a critical point. Then it means that if we have a region without a critical point, then it cannot have limiting cycles there.

**Proof** Let $C$ be a closed trajectory in $D$. Let $\Omega \subset D$ with $C = \delta\Omega$. Then we want to show that $\Omega$ has to contain a critical point. Suppose that there are not critical points on $\Omega$. Then we know that $F^2 + G^2$ everywhere in $\Omega$ because they cannot be zero simultaneously, since that would be a critical point. Now we measure the angle that the tangent vector along $C$ makes with a fixed direction as we move along $C$. Now since this is a closed curve, the angle must satisfy $\oint_C d\theta = 2\pi$. We know that the tangent of the angle is $\tan\theta = \frac{y'}{x'} = \frac{G}{F}$. Then we can write $d\theta = \frac{FdG - GdF}{F^2+G^2} \implies \frac{d\theta}{dt} = \frac{F\frac{dG}{dt} - G\frac{dF}{dt}}{F^2+G^2}$. We use Green's theorem to calculate this:

$$\oint_C d\theta = \oint_C \frac{FdG - GdF}{F^2 + G^2} = \iint_\Omega \frac{\partial}{\partial F}\left(\frac{F}{F^2 + G^2}\right) + \frac{\partial}{\partial F}\left(\frac{G}{F^2 + G^2}\right) dFdG$$

note that the function inside the integral is exact (and hence can be written as the derivative of some generating function), which would imply that the integral $\oint_C d\theta = 0$ because the value of the generating function at the end points will be exactly the same (since the curve is closed). But we required that $\oint_C d\theta = 2\pi$. Contradiction.

The reason why the contradiction works is that $F^2+G^2$ must vanish at some point so the potential function is not defined over the whole region and hence is not exact.

**Criterion 2: Bendixson's Criterion** If $F$ and $G$ have continuous partial derivatives in x and y in some region D and D is simply connected (i.e. no holes), then if $F_x+G_y$ does not change sign over the region D, then there are no closed orbits in D.

**Proof** Suppose, to the contrary, that there exists a closed trajectory $C$ in $D$. We apply Green's theorem to the closed trajectory $C$. Then we write:

$$\oint_C (F\vec{v}_1 + G\vec{v}_2) \cdot \vec{n}ds = \oint_C (Fdy - Gdx)$$

$$= \iint_\Omega \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dxdy \neq 0 \text{ by hypothesis that the sum of partial derivatives does not change sign}$$

But we note that the integral: $\oint_C (Fdy - Gdx)$ on a closed trajectory is the integral of the tangent vector of the curve dotted with the normal vector of the curve. But clearly, this dot product has to be zero because the tangent vector is always orthogonal to the normal vector. Hence we have a contradiction, and hence there cannot be any closed trajectories in the region D.

**Criterion 3** Let $F$ and $G$ have continuous partial derivatives in x and y in some region D. If there are no critical points in some bounded region $R \subset D$, then if there exists $t_0$ such that the solution $(x(t), y(t))$ stays in $R$ for all $t > t_0$, then the solution is either periodic or spirals towards a periodic solution. Then the region $R$ must contain a critical point or have a hole that contains a critical point.

If we have a trajectory $(x(t), y(t))$ that remains for all $t \geq t_0$ inside $R$, then there are only three possibilities:

- The trajectory approaches a critical point.
- The trajectory is a closed orbit.
- The trajectory approaches a closed orbit.

**Sketch of proof** Let $x(t_0) = x_0 \in R$. Then the trajectory passes through $x_0$. Pick a transversal curve $\sum$ that passes through $x_0$. Then the trajectory makes an angle with the transversal curve. Let $x_n$ be the points of subsequent return of the solution to $\sum$. Such points may or may not exist. If any of the return points $x_n$ is the same as a previous return point $x_k, k < n$, then the solution is a closed orbit. Now if the return points do not repeat, then it must be the case that the points $x_n$ proceed in one direction along $\sum$.

## 10.2 Wednesday 3 Dec 2014

**Review of previous session** Recall that we wanted to find trajectories that are solutions to the non-linear system:

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

on the plane. Let the first partial derivatives of $F$ and $G$ be continuous on region $R$. Then a trajectory $\vec{x}(t) = (x(t), y(t))$ that stays inside $R$ for all $t \geq t_0$ is either (i) approaching a critical point or (ii) a closed orbit or (iii) approaching a closed orbit.

**Proof** Let the trajectory pass through $\vec{x}(t_0) = \vec{x}_0$. Consider a curve $\Sigma$ that also passes through $\vec{x}_0$ and does not have a tangent vector that is coincident to the tangent vector of $\vec{x}$ at $\vec{x}_0$. We claim that the points $\vec{x}_n$, the nth intersection of the trajectory with $\Sigma$, follows monotonically the same order on $\Sigma$. This follows because the flow represented by $F$ and $G$ is continuous, hence at some point between two intersections we must have that the tangent vector of $\vec{x}$ turns around so that it can intersect $\Sigma$ again. An intersection cannot occur between two previous intersections since there would be some place where it is not traverse to the flow. ??? Since the sequence $\{\vec{x}_n\}$ is monotonic and bounded, it will have a limit at $\vec{y} \in \Sigma$. Now consider the solution that starts at $\vec{y}$ and look at where it first returns to $\Sigma$ again. Consider the first return map $\vec{x} \in \Sigma \mapsto \gamma(x) \in \Sigma$ which is the first point where the solution starting at $\vec{x}$ returns to $\Sigma$. Then we know that $\gamma(\vec{y}) = \lim_{n\to\infty} \gamma(\vec{x}_n) = \vec{y}$ by construction of $\vec{y}$. But the point whose return map is itself is a closed curve. Hence the orbit starting at $\vec{y}$ is a closed orbit.

**Damped Pendulum again** Recall that in the damped pendulum:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

In the undamped case, the energy function $V = (1 - \cos x) + \frac{y^2}{2\omega^2}$ is constant along the trajectory since using the chain rule $\frac{dV}{dt} = 0$. However, in the damped case $\gamma \neq 0$ and $\frac{dV}{dt} = -\frac{\gamma}{\omega^2} y^2 \leq 0$ the energy is decreasing along trajectories.

**Examining Energy Functions** Note that in the previous case, we know that the energy function is constantly decreasing along the trajectory. Hence we can examine the level sets of the energy functional to estimate the trajectories of the system. The trajectories will cut the level sets and tend towards the minimal energy areas.

**Lyapunov Stability** Consider a point $\vec{x}_0 = (x_0, y_0)$ which is a critical point for the non-linear system. Let $U(\vec{x}_0)$ be some open neighbourhood of the critical point. Consider a real-valued function defined on the neighbourhood: $L : U(\vec{x}_0) \to \mathbb{R}$

which is continuous on $U(\vec{x}_0)$ and satisfies $L(\vec{x}(t_0)) \geq L(\vec{x}(t_1))$ if $t_0 < t_1$, $\vec{x}(t_0), \vec{x}(t_1) \in U(\vec{x}_0)\backslash\{\vec{x}_0\}$. Call $L$ the Lyapunov function. Call this function a strict Lyapunov function if $L(\vec{x}(t_0)) > L(\vec{x}(t_1))$. If we can find such a strict Lyapunov function, then there are no closed orbits in $U(\vec{x}_0)$ because if there is a closed orbit then this function will have to keep decreasing and will hence achieve a different value when the trajectory returns to the starting point. **Sublevel sets of L** Consider the set $S_\delta = \{\vec{x} \in U(\vec{x}_0) | L(\vec{x}) \leq \delta\}$. Define $S^0_\delta \subset S_\delta$ that is the connected component of $S_\delta$ that contains $\vec{x}_0$. If $S^0_\delta \subset U(\vec{x}_0)$ is a closed set and has no boundary in common with the boundary of $\partial U(\vec{x}_0)$ then it is invariant under the forward time evolution. If you are inside the sub level set you remain inside the sub level set.

Now we can take $\delta$ to be small and obtain that the sub level set will shrink around $\vec{x}_0$. Formally, $\forall \epsilon > 0, \exists \delta > 0$ such that $S^0_\delta \subseteq B_\epsilon(\vec{x}_0)$ and $B_\delta(\vec{x}_0) \subseteq S^0_\epsilon$ where $B_\epsilon$ is the n-ball of radius $\epsilon$ entered at $\vec{x}_0$. This says that the $\{S^0_\delta\}$ are shrinking to $\vec{x}_0$ as $\delta \to 0$. **Proof** Consider the statement $S^0_\delta \subseteq B_\epsilon(\vec{x}_0)$. If this was false then $\forall n \in \mathbb{N}$, there exists $\vec{x}_0 \in S^0_{1/n}$ with $||\vec{x}_n - \vec{x}_0|| \geq \epsilon$. But since $S^0_{1/n}$ are connected sets we can follow the point outside the n-ball towards $\vec{x}_0$, and at some point we will cut the boundary of the n-ball. Formally, there exists $\vec{y}_n$ with $||\vec{y}_n - \vec{x}_0|| = \epsilon$. Then we have an infinite sequence of points $\vec{y}_n$ on the n-ball of radius $\epsilon$ around $\vec{x}_0$. This n-ball is a compact set and hence there has to be some subsequence that converges to a point on the circle $\vec{y}_{n_k} \to \vec{y} \in \partial B_\epsilon(\vec{x}_0)$. But the function $L$ is continuous hence we also have $L(\vec{y}_{n_k}) \to L(\vec{y})$. But we know that $\vec{y}_{n_k} \in S^0_{1/nk}$ and $L(\vec{y}_{n_k}) \leq \frac{1}{n_k} \to 0$ and hence $L(\vec{y}) = 0$. But we know that $\vec{y}$ is at a distance $\epsilon$ from the centre of the n-ball $\vec{x}_0$ and we know that the minimal value of L occurs at $\vec{x}_0$ Contradiction.

Now consider the second property $B_\delta(\vec{x}_0) \subseteq S^0_\epsilon$. If this were false then there exists $\vec{x}_n$ with $||\vec{x}_n - \vec{x}_0|| \leq \frac{1}{n}$ but with $L(\vec{x}_n) > \epsilon$. But $L$ is continuous so as $\vec{x}_n \to \vec{x}_0$ implies $L(\vec{x}_n) \to L(\vec{x}_0) = 0$. Contradiction.

## 10.3 4 Dec 2014 Recitation

**Exponential of commuting matrices** Prove that if $AB = BA$ then $\exp(A + B) = \exp(A)\exp(B)$.

**Proof** Write the LHS as:

$$\exp(A + B) = I + \frac{A + B}{1!} + \frac{(A + B)^2}{2!} + \ldots$$

Note that the binomial expansion is not true in general unless the matrices commute. But since this is the case here, we write:

$$\begin{aligned}\exp(A + B) &= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k}}{n!} \\ &= \exp(A)\exp(B)\end{aligned}$$

after some algebra.

## 10.4 Friday 5 Nov 2014

**Lyapunov Function** Recall that if $\vec{x}_0$ is a critical point of a non-linear equation and there is a Lyapunov function L in the neighbourhood $U(\vec{x}_0)$ of $\vec{x}_0$ such that $L : U(\vec{x}_0) \to \mathbb{R}$ is continuous and satisfies $L(\vec{x}_0) = 0$ and $L(\vec{x}) > 0$ and $L(\vec{x}(t_0) \geq L(\vec{x}(t_1))$ for $t_0 < t_1$ for a solution $\vec{x}(t)$ of the non-linear equation, then $\vec{x}_0$ is a stable critical point. If $L$ is strict, then the solutions $\vec{x}(t)$ flow into the critical point. Note that we have only defined $L$ to be continuous. If the strict Lyapunov function is differentiable, then $L(\vec{x}(t_0)) > L(\vec{x}(t_1))$ for $t_0 < t_1$ is equivalently written as $\frac{d}{dt}L(\vec{x}(t)) < 0$. We write:

$$\frac{d}{dt}L(\vec{x}(t)) = \nabla L(\vec{x}(t)) \cdot (F, G)$$

where we recall that the non-linear equation was $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$ so $(F, G)$ is the derivative of $\vec{x}(t)$ along the solution flow. Hence we can write the strict differentiable Lyapunov function condition as:

$$(\nabla L) \cdot (F, G) < 0$$

**Example** Consider the system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

Consider the function $L(x, y) = x^2 + y^2$. This is a differentiable function, hence we can check the condition for the Lyapunov function. We have that $\nabla L = (2x, 2y)$ and $(\nabla L) \cdot (y, -x) = (2x, 2y) \cdot (y, -x) = 2xy - 2yx = 0$. Hence this is a non-strict Lyapunov function and $L$ is constant along solution flows. Hence trajectories stay on level sets of $L$, and hence the solutions must be moving on circles around the origin (i.e. closed orbits) because $L(x, y)$ is radially symmetric.

**Example 2** Consider the system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x - \eta y$$

Consider the same function as above: $L(x, y) = x^2 + y^2$. Then we examine $\nabla L \cdot (F, G) = 2xy - 2yx - 2\eta y^2 = -2\eta y^2$. Then $L$ is a Lyapunov function and $\vec{x}_0 = (0, 0)$ is a critical point and the solutions spiral into the critical point.

**Summary of Lyapunov functions** For the linear system $\dot{x} = F(x, y), \dot{y} = G(x, y)$ with $(0, 0)$ being a critical point and if (1) there exists $L(x(t), y(t))$ with $\frac{dL}{dt} < 0$ then $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$. (2) If $\frac{dL}{dt} \leq 0$ then $(0, 0)$ is stable, including the possibility that $(0, 0)$ is a centre. (3) If $\frac{dL}{dt} > 0$ then $(0, 0)$ is an unstable critical point.

**Polar Coordinates Example** Consider the system:

$$\frac{dr}{dt} = r(1 - r^2) + \mu r \cos\theta$$
$$\frac{d\theta}{dt} = 1$$

where the $\mu > 0$ term represents a small perturbation. If $\mu = 0$ then we know that we have closed orbits with $r = 1$. Consider an annular region $R$ with inner radius $r_1$ and outer radius $r_2$. We choose the inner radius such that $r_1 < \sqrt{1 - \mu}$ and the outer radius large. We examine the behaviour of the solution along the boundaries of the annulus. Note that along the inner boundary, $\frac{dr}{dr} \sim r + \mu \cos\theta$. Along the outer boundary, the dominant term is nonlinear $\frac{dr}{dt} \sum -r^3$. Hence solutions flow into $r_2$. If $\mu$ is sufficiently small such that sign changes in the cosine do not affect the overall sign of $\frac{dr}{dt}$, then solutions at $r_1$ tend to flow outwards. Then there must be a closed orbit that remains in the annulus since solutions flow into $R$ and there are no solutions flowing out of $R$.

**Example: Van der Pol equation for triode oscillations** Consider the system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + \mu(1 - x^2)y$$

This system has a critical point at $(0, 0)$ and the linearisation there is:

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = -x + \mu y$$

Examining the trace and determinant, we obtain that if $0 < \mu < 2$, then we have an unstable spiral and if $\mu \geq 2$ then we have an unstable node. We observe that:

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \mu(1 - x^2)$$

hence in the region $|x| < 1$ the function above will be positive and hence there will not be any periodic orbits there. We now transform the system to polar coordinates. Writing $r^2 = x^2 + y^2$, we have that:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} = \mu(1 - r^2\cos^2\theta)r^2\sin^2\theta$$

Consider the annular region $R$ will small inner radius $r_1$ and large outer radius $r_2$. At the large $r_2$ boundary we know that the dominant term will be the $\frac{dr}{dt} = -\mu r^3\cos^2\theta\sin^2\theta$ (there is an r on the LHS) which is non-positive. Note that near the y-axis, this is no longer the dominant term and the sign of $\frac{dr}{dt}$ is positive. Hence the solutions may escape the region near the y-axis.