ACM95b Book Notes

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1. **Abel's Theorem**: If y_1 and y_2 are solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on the open interval I, then the Wronskian is given by:

$$W(y_1, y_2)(t) = ce^{-\int^t p(s)ds}$$

2. **Regular Singular Point**: Given $P(x)y'' + Q(x)y' + R(x)y = 0$, a regular singular point of the DE is a point x_0 such that $\lim_{x\to x_0}(x - x_0)\frac{Q(x)}{P(x)}$ is finite and $\lim_{x\to x_0}(x - x_0)^2\frac{R(x)}{P(x)}$ is finite as well.

3. **Reduction of order** Consider $y'' + p(t)y' + q(t)y = 0$. Suppose we know one solution $y_1(t)$. To find the second solution, let $y = v(t)y_1(t)$ and sub to obtain $y_1v'' + (2y_1' + py_1)v' = 0$ which is first order in v'.

4. **Variation of parameters** Consider $y'' + p(t)y' + q(t)y = g(t)$ with homogeneous solution $y_c(t) = c_1y_1(t) + c_2y_2(t)$. Then let c_1, c_2 vary with t and introduce the additional condition $c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$. Then solve to get $c_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt + c_3, c_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt + c_4$ so that the particular solution is

$$y_p(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}ds$$

5. **Variation of Parameters - nth order** Consider the system $y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = g(t)$ with homogenous solution $c_1y_1(t) + \ldots c_ny_n(t)$. Let the coefficients vary with t and impose the conditions $u_1'y_1^{(j)} + u_2'y_2^{(j)} + \ldots + u_n'y_n^{(j)} = 0, j = 0, 1, \ldots, n-2$. Then use Cramer's rule to write the particular solution as $y_p(t) = \sum_{m=1}^{n} y_m(t)\int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)}ds$ where W_m is the determinant obtained from the Wronskian matrix by replacing the mth column by $(0, 0, \ldots, 0, 1)$.

6. **Euler Equations** Consider $x^2y'' + \alpha xy' + \beta y = 0$. Making the substitution $y = x^r$, we obtain the indicial equation $F(r) = r(r-1) + \alpha r + \beta = 0$.
   - Real, distinct roots: $y = c_1|x|^{r_1} + c_2|x|^{r_2}$
   - Equal roots: $y = (c_1 + c_2 \ln|x|)|x|^{r_1}$
   - Complex roots $r = \lambda \pm i\mu : y = c_1|x|^{\lambda}\cos(\mu \ln|x|) + c_2|x|^{\lambda}\sin(\mu \ln|x|)$.

7. **Series solutions at ordinary points** Given:

$$y'' + \sum_{n=0}^{\infty} p_n(x - x_0)^n y' + \sum_{n=0}^{\infty} q_n(x - x_0)^n y = 0$$

   The recursion relation is:

$$(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}(n-k+1)p_ka_{n-k+1} + \sum_{k=0}^{n} q_ka_{n-k} = 0$$

8. **Singular Points** Given $P(x)y'' + Q(x)y' + R(x)y = 0$ and $P(x_0) = 0$ and at least one of $Q(x_0), R(x_0) \neq 0$, then x_0 is a singular point.
   - Regular Singular Point:

$$\lim_{x\to x_0}(x - x_0)\frac{Q(x)}{P(x)} \text{ is finite}$$
$$\lim_{x\to x_0}(x - x_0)^2\frac{R(x)}{P(x)} \text{ is finite}$$

- Regular singular point (higher order). Assume unity coefficient for $y^{(n)}$:

$$\begin{aligned}(x-x_0)^n p_0(x) &\text{ is finite}\\ (x-x_0)^{n-1} p_1(x) &\text{ is finite}\\ &\vdots\\ (x-x_0) p_{n-1}(x) &\text{ is finite}\end{aligned}$$

- Point at infinity. Make the substitution $x = \frac{1}{t}$ and change differentiation variables:

$$\frac{d}{dx} = -t^2 \frac{d}{dt}$$
$$\frac{d^2}{dx^2} = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}$$

  Then consider point at $t \to 0$.

9. **Series solutions near regular singular points**: Given $P(x)y'' + Q(x)y' + R(x)y = 0$ and a regular singular point $x_0 = 0$ (make change of variables to shift regular singular point to origin), write $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. Then we have:

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n$$
$$x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

on some interval containing the origin. We make the guess $y = x^r \sum_{n=0}^{\infty} a_n x^n$. This results in

$$a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] \right\} x^{r+n} = 0$$

Then we obtain the recurrence relation of a_n in terms of the $n-1$ previous terms:

$$F(r+n) a_n + \sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] = 0$$

and the zeroth order indicial equation ($a_0 \neq 0$):

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$
$$p_0 = \lim_{x \to 0} x \frac{Q(x)}{P(x)}$$
$$q_0 = \lim_{x \to 0} x^2 \frac{R(x)}{P(x)}$$

with roots r_1 and r_2.

- If $r_1 \neq r_2$ and they do not differ by an integer, then we have the solutions:

$$y_1 = |x|^{r_1} \left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right]$$
$$y_2 = |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} a_n(r_2) x^n \right]$$

- Complex roots do not differ by an integer. Hence just take the real and imaginary parts of the complex solution.

- Equal roots: The first solution is the same.
  From the result of the substitution:

$$a_0F(r)x^r + \sum_{n=1}^{\infty}\left\{F(r+n)a_n + \sum_{k=0}^{n-1}a_k[(r+k)p_{n-k}+q_{n-k}]\right\}x^{r+n} = 0$$

  and let r be a continuous variable. Then $a_n(r)$ be such that they solve the recurrence relation. In that case,

$$\mathcal{L}(r,x) = a_0F(r)x^r$$

  where $\mathcal{L}$ represents the operator for the homogeneous system. For repeated roots $F(r) = C(r-r_1)^2$. Differentiate and evaluate at $r = r_1$.

  The second solution can be found to be:

$$y_1 = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty}a_n(r_1)x^n\right]$$
$$y_2 = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty}a_n'(r_1)x^n$$

  where $a_n'(r_1)$ is $\frac{da_n}{dr}\big|_{r=r_1}$.
- Roots differing by integer: Pick the larger root r_2:

$$y_2 = ay_1(x)\ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty}c_n(r_2)x^n\right]$$
$$c_n(r_2) = \frac{d}{dr}[(r-r_2)a_n(r)]|_{r=r_2}$$
$$a = \lim_{r\to r_2}(r-r_2)a_N(r), N = r_1 - r_2 > 0$$

  Substitute this guess into the original equation to solve for a and $c_n(r_2)$.

10. **Bessel's Equation**: Consider $x^2y'' + xy' + (x^2 - v^2)y = 0$. $x = 0$ is a regular singular point.

- Bessel Equation of order Zero: Roots of Indicial equation are equal. Pick $v = 0$. Then the roots of the indicial equation are at 0. If we choose $a_1 = 0$ then we obtain the Bessel function of the first kind of order zero:

$$y_1(x) = a_0J_0(x) = a_0\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], x > 0$$

  Note that $J_0 \to 1$ as $x \to 0$. The second solution is given by:

$$y_2(x) = J_0\ln x + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}, x > 0$$

  where $H_m = \sum_{k=1}^{m}\frac{1}{k}$. We hence define the Bessel function of the second kind of order zero:

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2)J_0(x)\right]$$

  where γ is the Euler-Mascheroni constant given by:

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) \approx 0.5772$$

  Equivalently,

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}\right], x > 0$$

  Note that $Y_0(x)$ has a logarithmic singularity at $x = 0$. Then the general solution for the Bessel equation of order zero is:

$$y = c_1J_0(x) + c_2Y_0(x)$$

- Bessel equation of order one-half: Let $v = \frac{1}{2}$ so that the roots are $-\frac{1}{2}$ and $\frac{1}{2}$. The general solution is a linear combination of the Bessel functions of the first kind order of order plus or minus one-half:

$$y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$
$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$
$$J_{-1/2}(x) \left(\frac{2}{\pi x}\right)^{1/2} \cos x$$

- Bessel Equation of order one: Let $v = 1$. Then the roots of the indicial equation are -1 and 1.

11. **Laplace Transform**

- Sufficient condition for existence of Laplace transform: Suppose that f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A. Also assume that $|f(t)| \leq Ke^{at}$ when $t \geq M$ for positive constants K, M and real a. Then the Laplace transform $\int_0^\infty e^{-st} f(t)dt$ exists for $s > a$.
- Laplace Transform of derivatives: Suppose that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$, and that f is of exponential order as $t \to \infty$. Then:

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

- Higher order derivatives: Suppose that $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose that all of the $n-1$ derivatives are of exponential order (compared to Ke^{at}) for all $t \geq M > 0$. Then $\mathcal{L}(f^{(n)}(t))$ exists for $s > a$ and is given by:

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$$
$$\mathcal{L}(f^{(n)}(t)) = s^n\mathcal{L}(f(t)) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- Displacement: If $F(s) = \mathcal{L}(f(t))$ exists for $s > a$, and if $c > 0$, then:

$$\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)) = e^{-cs}F(s), s > a$$

- General solution to second order differential equation: Given $ay'' + by' + cy = g(t)$, obtain:

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_0' = G(s)$$
$$Y(s) = \Phi(s) + \Psi(s) \iff y(t) = \phi(t) + \psi(t)$$

where $\phi(t)$ is the solution to $ay'' + by' + cy = 0, y(0) = y_0, y'(0) = y_0'$ and $\psi(t)$ is the solution to $ay'' + by' + cy = g(t), y(0) = 0, y'(0) = 0$. Also define the transfer function $H(s) = \frac{1}{as^2+bs+c}$ so that $\Psi(s) = H(s)G(s)$. Then by the convolution theorem: $\psi(t) = \int_0^t h(t-\tau)g(\tau)d\tau$ with $h(t)$ being the solution to the initial value problem $ay'' + by' + cy = \delta(t), y(0) = 0, y'(0) = 0$. Notice that $h(t)$ here is the Green's function!

12. **Linear Operators** Consider a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$.

- Operator norm: $||T|| = \max_{|\mathbf{x}|\leq 1} |T(\mathbf{x})|$, where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ is the Euclidean norm.
  - $||T|| \geq 0, ||T|| = 0 \iff T = 0$
  - $||kT|| = |k|||T||, k \in \mathbb{R}$.
  - $||S + T|| \leq ||S|| + ||T||$.
  - $||T(\mathbf{x})|| \leq ||T|||\mathbf{x}|$.
  - $||TS|| \leq ||T||||S||$
  - $||T^k|| \leq ||T||^k, k = 0, 1, 2, \ldots$.
- Convergence: A sequence of linear operators $T_k \in L(\mathbb{R}^n)$ is said to converge to a linear operator $T \in L(\mathbb{R}^n)$ as $k \to \infty$: $\lim_{k\to\infty} T_k = T$ iff for all $\epsilon > 0$ there exists an N such that for $k \geq N, ||T - T_k|| < \epsilon$.
- Operator exponential convergence theorem: Given $T \in L(\mathbb{R}^n)$ and $t_0 > 0$, the series $\sum_{k=0}^\infty \frac{T^k t^k}{k!}$ is absolutely and uniformly convergent for all $|t| \leq t_0$.
- Operator exponential: $e^T = \sum_{k=0}^\infty \frac{T^k}{k!}$ is absolutely convergent.

13. **Systems of first order linear equations**

- Existence and Uniqueness: Suppose we have the system:

$$x_1' = F_1(t, x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$x_n' = F_n(t, x_1, x_2, \ldots, x_n)$$

  and let $F_i, i = 1, 2, \ldots, n$ and all the first order partial derivatives $\frac{\partial F_i}{\partial x_j}$ be continuous in a region R in $tx_1x_2\cdots x_n$ space with the initial conditions $(t_0, x_1^0, \ldots, x_n^0) \in R$. Then there exists an interval $|t - t_0| < h$ in which there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system that satisfies the initial conditions.

- Linear system of equations: Consider the system:

$$x_1' = p_{11}(t)x_1 + \ldots + p_{1n}(t)x_n + g_1(t)$$
$$\vdots$$
$$x_n' = p_{n1}(t)x_1 + \ldots + p_{nn}(t)x_n + g_n(t)$$

  that is linear in all x_i. If the functions p_{ij} and g_i are continuous on the interval $\alpha < t < \beta$ then there exists a unique solution that satisfies the initial conditions for $t_0 \in I = (\alpha, \beta)$. Moreover, the solution exists throughout I.

- Scalar inner product:

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \bar{y}_i$$

  for two column vectors $\mathbf{x}, \mathbf{y}$.

- Hermitian matrices: $\mathbf{A}^\dagger = \mathbf{A}$. Properties: Real eigenvalues, exactly n linearly independent eigenvectors, eigenvectors corresponding to different eigenvalues are orthogonal, within each eigenvalue of multiplicity m, it is possible to choose m mutually orthogonal eigenvectors. Hence the eigenvectors of $\mathbf{A}$ can be chosen to be orthogonal and linearly independent.

- Abel's Theorem for Linear Matrix systems: Given $\frac{d}{dt}\mathbf{x} = \mathbf{A}(t)\mathbf{x}$, the Wronskian can be determined up to a scaling constant by: $W = C \exp\left(\int_{t_0}^{t} Tr(\mathbf{A}(t'))dt'\right)$.

- Homogenous linear system with constant coefficients: Consider the matrix differential equation $\mathbf{x'} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is an $n \times n$ matrix. Consider solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, which gives the eigenvalue equation $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$. There are now the following possibilities:
  - Eigenvalues are real and distinct: General solution is $\mathbf{x} = c_1\boldsymbol{\xi}_1 e^{r_1 t} + \cdots + c_n \boldsymbol{\xi}_n e^{r_n t}$ for the n eigenvalues $r_1, \ldots, r_n$ with corresponding linearly independent eigenvectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$. If $\mathbf{A}$ is real and symmetric (special case of Hermitian), then we know that the eigenvalues will be real.
  - One set of complex eigenvalues. The complex eigenvalues and corresponding eigenvectors will be complex conjugates of each other (see by taking the complex conjugate of the eigenvalue equation). Let $\boldsymbol{\xi}_{1,2} = \mathbf{a} \pm i\mathbf{b}$ and $r_{1,2} = \lambda \pm i\mu$. Then the vectors:

$$\mathbf{u}(t) = e^{\lambda t}\left(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)\right)$$
$$\mathbf{v}(t) = e^{\lambda t}\left(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t)\right)$$

    are two real-valued solutions (taking the real and imaginary parts of the complex-valued $(\xi)e^{rt}$). Assume that all other eigenvalues are real and distinct. Then the general solution is $\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\boldsymbol{\xi}_3 e^{r_3 t} + \ldots + c_n \boldsymbol{\xi}_n e^{r_n t}$.

- **General solution to constant coefficient system**

$$\mathbf{x} = \sum_{j=1}^{k}\sum_{i=1}^{m_j} c_{ij} z^{i-1} \exp(\lambda_j z)$$

  For roots λ_j with multiplicities m_j such that $m_1 + m_2 + \ldots + m_k = n$, the order of the system.

- Fundamental matrix: Consider the matrix differential equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ with the fundamental set of solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$. Then the matrix:

$$\mathbf{\Psi}(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)]$$

is called the fundamental matrix for the system. The fundamental matrix is non-singular because the columns are linearly independent. Then we can write the general solution of the initial value problem as $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$ where $\mathbf{c}$ is the constant vector of coefficients $c_1, \ldots, c_n$. Then the initial condition can be written as $\mathbf{x}(t_0) = \mathbf{x}_0 = \mathbf{\Psi}(t)\mathbf{c}$ so $\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}_0 \implies \mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}_0$.

- Special Fundamental Matrix: Let $\mathbf{\Phi}(t)$ be the fundamental matrix that satisfies $\mathbf{\Phi}(t_0) = \mathbf{I}$. Its inverse at $t = t_0$ is clearly still the identity. Then the general solution can be written as $\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}_0$. Clearly, $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$.
- Matrix exponential: Define:

$$\exp(\mathbf{A}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

This matrix exponential clearly satisfies $\exp(\mathbf{A}t)|_{t=0} = \mathbf{I}$ and satisfies the initial value problem $\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}$, $\mathbf{\Phi}(0) = \mathbf{I}$. Hence the general solution is $e^{\mathbf{A}t}\mathbf{x}_0$. More generally, for a non-homogeneous system,

$$\mathbf{x} = \exp(\mathbf{A}z)\mathbf{x}_0 + \int_0^z \exp(\mathbf{A}(z-t))\mathbf{f}(t)dt$$

For a **general fundamental solution**:

$$\mathbf{x} = \Psi(z)\Psi^{-1}(0)\mathbf{x}_0 + \Psi(z)\int_0^z \Psi^{-1}(t)\mathbf{f}(t)dt$$

- Obtaining the Matrix Exponential. Consider $e^{\mathbf{A}t}$. Let $\mathbf{A}$ be diagonalized by $\mathbf{T}$: $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Then:

$$e^{\mathbf{A}t} = e^{\mathbf{T}\mathbf{D}\mathbf{T}^{-1}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$$

(show this by Taylor expansion).

- Diagonalisable matrix. A matrix $\mathbf{A}$ is diagonalisable if it is similar to a diagonal matrix $\mathbf{D}$. That is, there exists an invertible matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$. If $\mathbf{A}$ has less than n linearly independent eigenvectors, then it is not diagonalisable. If $\mathbf{A}$ is Hermitian, then its eigenvectors are mutually orthogonal, and can be normalised such that the inner product $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_i) = 1, \forall i = 1, 2, \ldots, n$. It can be shown that in such as case $\mathbf{T}^{-1} = \mathbf{T}^*$ so $\mathbf{T}$ is Hermitian as well.
- **Repeated Eigenvalues**: Note that there are two cases: either there are k linearly independent eigenvectors corresponding to the eigenvalue of algebraic multiplicity k (i.e. geometric multiplicity k) or there are fewer than k linearly independent eigenvectors (geometric multiplicity less than k).
  - In the first case, tack on the exponential e^{rt} and each of the eigenvectors becomes a linearly independent solution to the system: $\boldsymbol{\xi}_1 e^{rt}, \ldots, \boldsymbol{\xi}_k e^{rt}$. This first case always occurs when the coefficient matrix is Hermitian.
  - In the second case, let eigenvalue have algebraic multiplicity two and geometric multiplicity one. Let the eigenvalue be r and the eigenvector be $\boldsymbol{\xi}$. Then solve for the vector $\boldsymbol{\eta}$ that satisfies $(\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$. Then the second solution is $\mathbf{x}_2 = \boldsymbol{\xi}te^{rt} + \boldsymbol{\eta}e^{rt}$. $\boldsymbol{\eta}$ is a generalised eigenvector corresponding to the eigenvalue r.

14. **Linear Inhomogeneous Matrix Differential Equation (Finding particular solution with n eigenvectors)**. Consider $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ with $\mathbf{A}$ is a constant matrix with n linearly independent eigenvectors. Construct the matrix of eigenvectors $\mathbf{T}$ and introduce the change of variable $\mathbf{x} = \mathbf{T}\mathbf{y}$ and substitute to obtain $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ where $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$. Then we have the decoupled equations:

$$y_j'(t) = r_j y_j(t) + h_j(t), j = 1, 2, \ldots, n$$

in terms of the eigenvalues r_j. Integrating, we obtain:

$$y_j(t) = e^{r_j t}\int_{t_0}^t e^{-r_j s}h_j(s)ds + c_j e^{r_j t}, j = 1, 2, \ldots, n$$

for constants c_j so that they can be absorbed into the homogeneous solutions $e^{r_j t}$. Then the particular solution is:

$$y_j(t) = e^{r_j t}\int_{t_0}^t e^{-r_j s}h_j(s)ds, j = 1, 2, \ldots, n$$

15. **Linear Inhomogeneous Matrix Differential Equation (Defective Matrix)** If $\mathbf{A}$ is not diagonalisable, then we reduce it to Jordan form instead. The equations are not totally uncoupled but can still be solved consecutively starting from $y_n(t)$.

16. **Undetermined Coefficients for Systems:** Consider $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$. If $\mathbf{A}$ is a constant coefficient matrix, and $\mathbf{g}$ is a polynomial, exponential or sinusoidal function (or a composition of these), then make the following guesses:

    - $\mathbf{g}(t) = t\mathbf{a} + \mathbf{b}$. Make the guess $t\mathbf{c} + \mathbf{d}$.
    - $\mathbf{g}(t) = \mathbf{u}e^{\lambda t}$ when λ is a simple root of the characteristic equation. Assume solution of form $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$.

17. **Variation of Parameters (coefficient matrix not constant or not diagonalisable)** Consider $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}+\mathbf{g}(t)$. Assume that we can find the fundamental matrix for the homogenous system: $\mathbf{\Psi}(t)$, which is non-singular on any interval when $\mathbf{P}(t)$ is continuous. We seek a solution of the non homogenous system in the form: $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ for some vector $\mathbf{u}(t)$ to be found. Then:

$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds + \mathbf{c}$$

$$\implies \mathbf{x} = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds + \mathbf{\Psi}(t)\mathbf{c}$$

    If the initial values are given:

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}_0 + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds$$

18. **Method of Adjoint Solution** Let $\mathbf{x}' = A(z)\mathbf{x} + f(z)$. Define the adjoint system:

$$\mathbf{y}' = -A^T(z)\mathbf{y}$$

    Define the matrix

$$\Phi(z) = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

    Where the solutions to the adjoint equation are in **rows**. Then the solution is:

$$\mathbf{x} = \Phi^{-1}(z)\Phi(z_0)\mathbf{x}_0 + \Phi^{-1}(z)\int_{z_0}^{z} \Phi(t)f(t)dt$$

    The adjoint fundamental matrix solves the equation: $\Phi' = -\Phi A$. Note that the inverse of the adjoint matrix is the fundamental matrix!

19. **Laplace Transform of Linear Systems** Transform of a vector is computed component by component. Then we have the following identities, taking $\mathbf{X}(s) = \mathcal{L}(\mathbf{x}(t))$:

$$\mathcal{L}(\mathbf{x}'(t)) = s\mathbf{X}(s) - \mathbf{x}(0)$$

20. **Boundary Value Problems** If the boundary value function has value zero for each $\mathbf{x}$, then the problem is homogeneous. Otherwise it is non-homogeneous.

21. **Fourier methods**

- Discrete series:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

- Coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$
$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

- Sufficient conditions for Fourier series convergence: Suppose f and f' are piecewise continuous on the interval $-L \leq x < L$. Suppose that f is defined outside the interval $-L \leq x < L$ so that it is periodic with period $2L$. Then f has a Fourier series whose coefficients can be calculated using the formulae above. The Fourier series converges to $f(x)$ at all points where f continuous, and it converges to $\frac{f(x_+)+f(x_-)}{2}$ at all points where f is discontinuous, where $f(x_+)$ is the limit of $f(x)$ from the right and $f(x_-)$ is the limit from the left.

22. **Even and odd function properties**

    - Sum (difference) and Product (quotient) of two even functions are even.
    - Sum (difference) of two odd functions is odd. Product (quotient) of two odd functions is even.
    - Sum (difference) of an odd function and an even function is neither even nor odd. Product (quotient) of an odd function and an even function is odd.

23. **Table of Laplace Transforms**

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$t^p, p > -1$	$\frac{\Gamma(p-1)}{s^{p+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$t^{n-1/2}, n = 1, 2, 3 \ldots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+1/2}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin(at+b)$	$\frac{s\sin b + a\cos b}{s^2+a^2}$
$\cos(at+b)$	$\frac{s\cos b - a\sin b}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f(ct)$	$\frac{1}{c}F(s/c)$
$u(t-c)$	$\frac{e^{-cs}}{s}$
$\delta(t-c)$	e^{-cs}
$u(t-c)f(t-c)$	$e^{-cs}F(s)$
$u(t-c)g(t)$	$e^{-cs}\mathcal{L}(g(t+c))$
$e^{ct}f(t)$	$F(s-c)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(u)du$
$\int_0^t f(v)dv$	$\frac{F(s)}{s}$
$f * g = \int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - \sum_{i=0}^{n-1} s^{n-1-i}f^{(i)}(0)$

Reduction of order $y''+p(x)y'+q(x)y=0$ with y_1 known. Then $y_2 = v(x)y_1$ and solve for v in $v'(2y_1'+py_1)+v''y_1=0$.

$$v(x) = c\int^x \frac{1}{y_1^2(t)}\exp\left[-\int^t p(s)ds\right]dt$$
$$y_{gen} = c_1y_1 + c_2v(x)y_1$$

Variation of Parameters Let $y_p = u_1y_1+u_2y_2$. Then set $u_1'y_1+u_2'y_2=0$. Then:

$$y_p = c_1y_1 + c_2y_2 - y_1\int^x \frac{y_2(t)r(t)}{W(t)} + y_2\int^x \frac{y_1(t)r(t)}{W(t)}$$

Systems of Equations $\frac{d\mathbf{x}}{dz} = \mathbf{A}(z)\mathbf{x}+\mathbf{F}(z)$.

Reduction of order for systems Assume we know $\mathbf{x}_1$ is a solution to $\mathbf{x}'=\mathbf{A}\mathbf{x}$. Define Γ to be the matrix with the last column of $\mathbf{x}_1$ and the last element of $\mathbf{x}_1$ must be nonzero. Then solve:

$$\mathbf{x} \equiv \Gamma\mathbf{y}$$
$$\mathbf{y}' = \Gamma^{-1}(A\Gamma-\Gamma')\mathbf{y} = B\mathbf{y}$$
$$y_i' = \sum_{j=1}^{n-1} b_{ij}y_j, i=1,2,\ldots,n-1$$
$$y_n' = \sum_{j=1}^{n-1} b_{nj}y_j$$

Adjoint System Given $\mathbf{x}'=\mathbf{A}\mathbf{x}+\mathbf{f}$, the adjoint system is $\mathbf{y}'=-\mathbf{A}^T\mathbf{y}$. Put the solutions to y in rows in Φ, which satisfies $\Phi'=-\Phi\mathbf{A}$. Then:

$$\mathbf{x} = \Phi^{-1}(z)\Phi(z_0)\mathbf{x}_0 + \Phi^{-1}(z)\int_{z_0}^z \Phi(t)\mathbf{f}(t)dt$$

Fundamental matrix solution Suppose you have the fundamental matrix Ψ. Then for the inhomogeneous system,

$$\mathbf{x} = \Psi(z)\Psi(z_0)^{-1}\mathbf{x}_0 + \Psi(z)\int_{z_0}^z \Psi^{-1}(t)\mathbf{f}(t)dt$$

Series solution Write a_n in terms of a_0!!! and check for series convergence

Cauchy Product

$$\sum_{n=0}^\infty a_n \sum_{n=0}^\infty b_n = \sum_{n=0}^\infty\sum_{j=0}^n a_jb_{n-j}$$

Point at infinity Let $x=\frac{1}{t}$. Then:

$$\frac{d}{dx} = -t^2\frac{d}{dt}$$
$$\frac{d^2}{dx^2} = t^4\frac{d^2}{dt^2}+2t^3\frac{d}{dt}$$

Laplace Transform of Derivative

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0)\ldots sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds$$

where c is chosen so that all the poles are to the left. Close in the direction that e^{st} does not blow up.

Convolution for Laplace Transform

$$f*g = \int_0^t f(t-\tau)g(\tau)d\tau$$

Sturm-Liouville ODE

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y(x) = 0$$
$$c_1y(a)+c_2y'(a)=0$$
$$d_1y(b)+d_2y'(b)=0$$
$$p(x)>0, r(x)>0$$
$$p,q,r\in C$$

Legendre Equation

$$(1-x^2)y''-2xy'+v(v+1)y=0$$
$$[(1-x^2)y']'+v(v+1)y=0, -1\le x\le 1$$

Lagrange Identity Define:

$$L[y(x)] = -\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x)$$

Then with separated homogeneous boundary conditions,

$$\int_a^b L[u(x)]v(x)dx = \int_a^b u(x)L[v(x)]dx$$
$$\int_a^b \bar{u}(x)L[v(x)]dx = \int_a^b v(x)L[\bar{u}(x)]dx$$

Fourier Series

$$f(x) = \sum_{n=1}^\infty A_n\sin(n\pi x) \iff A_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$
$$f(x) = \sum_{n=0}^\infty B_n\sin(n\pi x) \iff B_n = 2\int_0^1 f(x)\cos(n\pi x)dx$$
$$f(x) = \sum_{n=1}^\infty A_n\sin(2n\pi x) + \sum_{n=0}^\infty B_n\cos(2n\pi x)$$
$$\iff A_n = 2\int_0^1 f(x)\sin(2n\pi x)dx$$
$$\& \quad B_n = 2\int_0^1 f(x)\cos(2n\pi x)dx$$

Full Fourier Series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L) + \sum_{n=0}^{\infty} B_n \cos(n\pi x/L), -L < x < L$$

$$\iff A_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx$$

$$B_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L}$$

$$C_n = \begin{cases} \frac{B_n - iA_n}{2}, n > 0 \\ \frac{B_n + iA_n}{2}, n < 0 \\ B_0, n = 0 \end{cases}$$

Parseval's Theorem

$$\int_{-L}^{L} f(x)^2 dx = L\left[2B_0^2 + \sum_{n=1}^{\infty}(A_n^2 + B_n^2)\right]$$

Uniform Convergence of Fourier Series The Fourier series of $f(x)$ converges uniformly to $f(x)$ where $f(x)$ is differentiable. If $f(x)$ has a discontinuity at $x = a$, then the series converges to:

$$F(a) = \frac{f(a^+) + f(a^-)}{2}$$

and the convergence is not uniform.

Rate of convergence of Fourier Series If you have $k-1$ continuous derivatives including the zeroth derivative, the coefficients decay at least as fast as $\frac{1}{n^k}$.

Riemann-Lebesgue Lemma If $g(x)$ is any L_1 integrable function, then $\int_0^{2\pi} g(x)e^{ikx}dx \to 0$ as $k \to \infty$.

Inhomogeneous S-L Form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y(x) = r(x)f(x)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{f_n}{\lambda - \lambda_n}\phi_n(x)$$

Fredholm Alternative If $\lambda = \lambda_m$ and f_m vanishes,

$$y(x) = \sum_{n=0, n\neq m}^{\infty} \frac{f_n}{\lambda - \lambda_n}\phi_n(x) + C\phi_m(x)$$

without uniqueness.

Inhomogeneous boundary conditions Let $y(0) = y_0, y(L) = y_1$. Then define:

$$u(x) = y(x) - \frac{1}{L}\left[y_1 x + y_0(L-x)\right]$$

Green's functions

$$G(x, x'; \lambda) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n}$$

$$y(x) = \int_a^b f(x')r(x')G(x, x'; \lambda)dx'$$

$$\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) - q(x)G + \lambda r(x)G = \delta(x - x'), a < x < b$$

and it satisfies the homogeneous boundary conditions (clearly).

Fourier Transform

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

Fourier Transform of Derivative Requires the function and its derivative to vanish at infinity.

$$\mathcal{F}[y^{(n)}] = (ik)^n Y(k)$$

Convolution for Fourier Transform

$$f * g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\zeta)g(x - \zeta)d\zeta$$

Parseval's Theorem (Transform)

$$\int_{-\infty}^{\infty} |F(k)|^2 dk = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Sine and Cosine Transforms

$$F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos kx dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(k)\cos kx dk$$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} y''(x)\cos kx dx = -k^2 Y_c(k) - \sqrt{\frac{2}{\pi}}y'(0)$$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} y''(x)\sin kx dx = -k^2 Y_s(k) + \sqrt{\frac{2}{\pi}}y(0)$$

$f(x)$	$F(k) = \frac{1}{\sqrt{2\pi}} f(x) e^{-ikx} dx$
$f(x-a)$	$e^{-ika}F(k)$
$e^{iax}f(x)$	$F(k-a)$
$f(ax)$	$\frac{1}{\lvert a\rvert} F\left(\frac{k}{a}\right)$
$f^{(n)}(x)$	$(ik)^n F(k)$
$x^n f(x)$	$i^n \frac{d^n F(k)}{dk^n}$
Purely real, even	Purely real transform
Purely real odd	Purely imaginary transform
$\overline{f(x)}$	$\overline{F(-k)}$
1	$\sqrt{2\pi}\delta(k)$
$\delta(x)$	$\frac{1}{\sqrt{2\pi}}$
e^{iax}	$\sqrt{2\pi}\delta(k-a)$
$\cos(ax)$	$\sqrt{\frac{\pi}{2}}(\delta(k-a)+\delta(k-a))$
$\sin(ax)$	$\sqrt{\frac{\pi}{2}}\frac{1}{i}(\delta(k-a)-\delta(k+a))$
x^n	$i^n\sqrt{2\pi}\delta^{(n)}(k)$
$\frac{1}{x}$	$-i\sqrt{\frac{\pi}{2}}sgn(k)$
$sgn(x)$	$\sqrt{\frac{2}{\pi}}\frac{1}{ik}$
$u(x)$	$\sqrt{\frac{\pi}{2}}\left(\frac{1}{i\pi k}+\delta(k)\right)$

For finding Sturm-Liouville form
Recall that the Sturm-Liouville equation was of the form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y(x) = 0$$

Expanding,

$$p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} - q(x)y(x) + \lambda r(x)y(x) = 0$$

and hence comparing,

$$p(x) =$$
$$p'(x) =$$
$$q(x) =$$
$$r(x) =$$

We hence have the differential equation for $p(x)$ which we can separate variables and solve to obtain:

$$\int \frac{p'(x)}{p(x)}dx = \int dx \frac{a(x)}{b(x)}$$