ACM95c Book Notes

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Implicit Function Theorem (Wikipedia) Let $F(x,y)$ be defined on an open disk containing (a,b) where $F(a,b) = 0, F_y(a,b) \neq 0$ and F_x, F_y are continuous on the disk. Then $F(x,y) = 0$ defines y as a function of x near that point, and the derivative of that function is $\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$.

Definition 2.1.1: Integral Curve A curve C in Ω is an integral curve of the vector field $\boldsymbol{V}$ if $\boldsymbol{V}$ is tangent to C at each of its points.

Definition 2.1.2: Functionally independent Two functions $u_1(x,y,z)$ and $u_2(x,y,z)$ that satisfy $\nabla u_1 \times \nabla u_2 \neq \mathbf{0}, (x,y,z) \in \Omega$ are called functionally independent in Ω.

Notation Parametric form: Write $x' = P(x,y,z), y' = Q(x,y,z), z' = R(x,y,z)$. Equivalent Non-parametric form: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Equate any two of the equalities to solve for two variables in terms of a third (ensure that the function in the denominator is nonvanishing).

Definition 2.1.3 First Integral A function $u \in C^1(\Omega)$ is first integral of the vector field $\boldsymbol{V} = (P,Q,R)$ or $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ in Ω if at each point of Ω, $\boldsymbol{V}$ is orthogonal to ∇u. That is:

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0, \quad (x,y,z) \in \Omega$$

Any solution of the above PDE is a first integral of $\boldsymbol{V}$.

Theorem 2.1.1 Let u_1, u_2 be any two functionally independent first integrals of $\boldsymbol{V}$ in Ω. Then the equations $u_1(x,y,z) = c_1, u_2(x,y,z) = c_2$ describe the collection of all integral curves of $\boldsymbol{V}$ in Ω. This is a two-parameter family of curves in Ω.

Theorem 2.1.2 (a) If $f(u)$ is a C^1 function of a single variable u and if $u(x,y,z)$ is a first integral of $\boldsymbol{V}$, then $w(x,y,z) = f(u(x,y,z))$ is also a first integral of $\boldsymbol{V}$. (b) If $f(u,v)$ is a C^1 function of two variables and if $u(x,y,z)$ and $v(x,y,z)$ are any two first integrals of $\boldsymbol{V}$ then $w(x,y,z) = f(u(x,y,z), v(x,y,z))$ is also a first integral of $\boldsymbol{V}$.

Non-parametric form to find first integrals $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{d(ax+by+cz)}{aP+bQ+cR}$, where a,b,c are chosen so that the RHS can be written as some $\frac{df(x,y,z)}{f(x,y,z)}$ and be separable.

The general problem $Pu_x + Qu_y + Ru_z = 0$ with P,Q,R not vanishing simultaneously at any point of Ω.

Theorem 2.3.1: Composition of solutions Let u_1 and u_2 be two solutions of the general problem in Ω and let $F(u_1,u_2)$ be any C^1 function of two variables. Then $u(x,y,z) = F(u_1(x,y,z), u_2(x,y,z))$ is also a solution to the problem in Ω.

Theorem 2.3.2 Let u_1 and u_2 be two functionally independent solutions of the general problem in Ω and let (x_0,y_0,z_0) be any point of Ω. Let $u(x,y,z)$ be any solution of the general problem in Ω. Then there is an open neighbourhood $U \in \Omega$ of (x_0,y_0,z_0) and a C^1 function $F(u_1,u_2)$ of two variables such that $u(x,y,z) = F(u_1(x,y,z), u_2(x,y,z))$ for $(x,y,z) \in U$.

Definition 2.4.1: Integral Surface A surface $S \in \Omega \subset \mathbb{R}^3$ is an integral surface of the vector field $\boldsymbol{V}$ if S is a level surface of a first integral of $\boldsymbol{V}$. In other words, S is described by an equation of the form $u(x,y,z) = c$ where u is a solution to the equation $Pu_x + Qu_y + Ru_z = 0$ in Ω such that $\nabla u \neq \mathbf{0}$ in Ω. An integral surface of $\boldsymbol{V}$ is called a solution surface of $Pu_x + Qu_y + Ru_z$.

Theorem 2.4.1 If S is a solution surface of $Pu_x + Qu_y + Ru_z$ in Ω, then for every point of S, the solution curve of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ passing through that point lies in S. Conversely, if u is a C^1 function in Ω, and if at each point $(x_0,y_0,z_0) \in \Omega$ the solution curve of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ passing through (x_0,y_0,z_0) lies on the level surface of u passing through (x_0,y_0,z_0), then u is a solution of $Pu_x + Qu_y + Ru_z$ in Ω.

Theorem 2.4.2: Local integral surface existence, curve not an integral curve Let curve C be described parametrically by C^1 functions $x_0(t), y_0(t), z_0(t)$ with derivatives that do not vanish simultaneously on the interval. Suppose that $V = (P,Q,R)$ is not tangent to curve C at (x_0,y_0,z_0), which lies on the curve. Then in some neighbourhood Ω_0 of (x_0,y_0,z_0), there is a unique integral surface of V containing the part of C in Ω_0.

Finding an integral surface that contains a particular curve (not part of an integral curve) :

1. Express the curve parametrically in the neighbourhood of the point in terms of t.
2. Calculate the gradients of the curve in terms of t,
3. Calculate the vector field direction on the curve C, and show that it is nowhere tangent to C in some neighbourhood of the point.
4. Identify two functionally independent first integrals of $\boldsymbol{V}$ and express them parametrically in terms of t on the curve C.
5. Eliminate t by combining the equations for the two first integrals to obtain a single equation in terms of the first integrals.
6. Express the first integrals in (x, y, z) to obtain the integral surface.

Theorem 2.4.3 Finding an integral surface when the required curve is part of an integral curve Suppose that $\boldsymbol{V} = (P, Q, R)$ is tangent to C at one of its points contained in a neighbourhood Ω_0 of a point $(x_0, y_0, z_0) \in C$. Then in some neighbourhood $\Omega_1 \subset \Omega_0$ of that point there are infinitely many integral surfaces of $\boldsymbol{V}$ containing the part of C in Ω_1.

Theorem 2.5.1 Vector potential existence Let $\boldsymbol{V} = (P, Q, R)$ be a non vanishing vector field defined on $\Omega \subset \mathbb{R}^3$ with $P, Q, R \in C^1$. If $\boldsymbol{V}$ is solenoidal in Ω, which means that $\nabla \cdot \boldsymbol{V} = 0$, then given any point $(x_0, y_0, z_0) \in \Omega$, there is a neighbourhood Ω_0 of that point and a vector field $\boldsymbol{W}$ with C^1 components such that $\boldsymbol{V} = \nabla \times \boldsymbol{W}(x, y, z), (x, y, z) \in \Omega_0$. $\boldsymbol{W}$ is called a vector potential for the given field $\boldsymbol{V}$.

Vector Calculus Identities

$$\nabla \cdot (f\boldsymbol{u}) = \nabla f \cdot \boldsymbol{u} + f \nabla \cdot \boldsymbol{u}$$
$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v} - (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{u}$$
$$\nabla \times (\nabla f) = 0$$
$$\nabla \times (f\boldsymbol{u}) = (\nabla f) \times \boldsymbol{u} + f \nabla \times \boldsymbol{u}$$
$$\nabla G(u_1, u_2) = \frac{\partial G}{\partial u_1} \nabla u_1 + \frac{\partial G}{\partial u_2} \nabla u_2$$

Calculating the vector potential Define $\lambda(x, y, z)$ such that $\boldsymbol{V}(x, y, z) = \lambda(x, y, z) \nabla u_1 \times \nabla u_2$ because the two gradients of the first integrals must be perpendicular to the vector field, hence their cross product will be parallel to it. Then write it in terms of the first integrals $\lambda(x, y, z) = F(u_1, u_2)$. Find a function $G(x, y, z)$ such that $F(u_1, u_2) = \frac{\partial G}{\partial u_1}(u_1, u_2)$. Then $\boldsymbol{W} = G \nabla u_2$, and $\boldsymbol{V} = \nabla \times \boldsymbol{W}$.

Definition: Quasi-linear An equation of the form $P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z)$ is called quasi-linear.

Definition: Almost-linear An equation of the form $P(x, y)z_x + Q(x, y)z_y = R(x, y, z)$ is called almost linear.

Definition: Linear An equation of the form $a(x, y)z_x + b(x, y)z_y + c(x, y)z = d(x, y)$ is called linear.

Lemma 3.2.1 Solution to Quasi-linear form Consider the quasi-linear equation $P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z)$. Let u be in C^1 in $\bar{\Omega}$, the domain of the functions P, Q, R. Suppose that at every point we have that (i) $Pu_x + Qu_y + Ru_z = 0$, (ii) $u_z \neq 0$. Then $u(x, y, z) = 0$ implicitly defines z as a function of x, y and this function $z(x, y)$ satisfies the quasi-linear equation.

Theorem 3.2.1 Solution to quasi-linear form Let u_1 and u_2 be two functionally independent solutions of $Pu_x + Qu_y + Ru_z = 0$ in a domain $\bar{\Omega}$ of $\mathbb{R}^3$. Let $F(u_1, u_2)$ be an arbitrary C^1 function and consider the level surface $F(u_1(x, y, z), u_2(x, y, z)) = 0$. Every part of this surface having a normal with non-zero z-component defines z implicitly as a function of x and y, and this function is a solution of the quasi-linear equation $P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z)$. If u_1 is independent of z, then the general integral can be written as $u_2(x, y, z) = f(u_1(x, y))$ for an arbitrary function f of a single variable.

Definition 3.2.1: General Integral $F(u_1(x, y, z), u_2(x, y, z)) = 0$ is called a general integral of $P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z)$ in $\bar{\Omega}$, where u_1, u_2 are two functionally independent solutions of $Pu_x + Qu_y + Ru_z = 0$. Note that it is not the general solution of the quasi-linear equation.

Cauchy Problem Require a solution $z(x, y)$ to $P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z)$ which has given values on a given curve C in $\mathbb{R}^2$, where C is a 2D curve parametrised by $(x_0(t), y_0(t))$. A solution is a function $z(x, y)$ that satisfies the quasi-linear equation and that is equal to a given function $z_0(t)$ when described parametrically on an interval: $z(x_0(t), y_0(t)) = z_0(t), t \in I$. The given curve C is called the initial curve and $z_0(t)$ is the initial data.

Theorem 3.3.1 Existence and uniqueness of solution to Cauchy problem Suppose $P, Q, R \in C^1$ in domain $\bar{\Omega}$ containing (x_0, y_0, z_0). Let C be parametrized by $x = x_0(t), y = y_0(t), z = z_0(t), t \in I$. Let t_0 be such that $(x_0(t_0), y_0(t_0), z_0(t_0)) = (x_0, y_0, z_0)$. Suppose that $P(x_0, y_0, z_0)\frac{dy_0(t_0)}{dt} - Q(x_0, y_0, z_0)\frac{dx_0(t_0)}{dt} \neq 0$. Then there exists a unique solution of the Cauchy Problem satisfying the initial condition at every point of C contained in a neighbourhood of (x_0, y_0).

Corollary 3.3.1 Existence and Uniqueness for curves along axis Let $P, R \in C^1$ in $\mathbb{R}^3$ and $f \in C^1$ in $\mathbb{R}$. Then for the following problem: $P(x, y, z)z_x + z_y = R(x, y, z), z(x, 0) = f(x)$, that is, the solution is required to be equal to $f(x)$ along the x-axis. Then in the neighbourhood of every point in the x-axis, there is a unique solution of the problem.

Theorem 3.4.1 Nonexistence condition Suppose:

$$P(x_0, y_0, z_0)\frac{dy_0(t_0)}{dt} - Q(x_0, y_0, z_0)\frac{dx_0(t_0)}{dt} = 0 \iff \frac{\frac{dx_0(t_0)}{dt}}{P(x_0, y_0, z_0)} = \frac{\frac{dx_0(t_0)}{dt}}{Q(x_0, y_0, z_0)} = \mu$$

which is the negation of the condition required for Theorem 3.3.1. Now suppose: $\frac{\frac{dz_0(t_0)}{dt}}{R(x_0, y_0, z_0)} \neq \mu$. Then there is no solution to the initial value problem $P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z)$ for the initial condition curve C in any neighbourhood of the point (x_0, y_0).

Theorem 3.4.2 Non-uniqueness when the curve is an integral curve Let $\boldsymbol{T}(t) = (x_0'(t), y_0'(t), z_0'(t)), t \in I$, the tangent vector to the curve C parametrised by $x_0(t), y_0(t), z_0(t)$. Suppose that $\boldsymbol{T}(t_0) = \mu(t)\boldsymbol{V}, t \in I$, so that the two quantities are proportional at every point of C near (x_0, y_0, z_0). Then the initial-value problem has infinitely many solutions in a neighbourhood of (x_0, y_0) since every integral surface passing through that point will contain the integral curve.

IVP for conservation laws Consider the system $a(z)z_x + z_y = 0, z(x, 0) = f(x)$ where $a, f \in C^1$. The associated system of ODEs can be written as $\frac{dx}{a(z)} = \frac{dy}{1} = \frac{dz}{0}$ with two functionally independent first integrals as $u_1 = z, u_2 = x - a(z)y$. Hence the general integral is $z = f(x - a(z)y)$, where we include the boundary condition $z(x, 0) = f(x)$. This solution exists as long as $1 + f'(x - a(z)y)a'(z)y > 0$ (the denominator of z_x, z_y is non-zero and positive).

Probability generating function $G(t, s) = \sum_{n=0}^{\infty} P_n(t)s^n$.

Stopped at Page 100, Chapter 4 Cauchy-Kovalevsky Theorem

Principal part Highest order terms appearing in the equation. Denote this operator as $P_m(x, D)$, where m represents the highest order, x is the independent variable, D represents the differentiation operator.

Characteristic direction at a point $x \in \mathbb{R}^n$ is a non-zero vector (made of numbers!) $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$ such that $P_m(x, \zeta) = 0$. This is the characteristic equation. Basically, replace all D_1 with ζ_1, etc.

Characteristic Surface The surface S is said to be characteristic at x^0 with respect to $P(x, D)$ if a vector normal to S at x^0 defines an unsigned direction which is characteristic with respect to $P(x, D)$ at x^0. If S is characteristic with respect to $P(x, D)$ for all of its points, it is a characteristic surface. In $\mathbb{R}^2$, it is a characteristic curve. Note that the normal vector to a parametrised curve $(x_0(t), y_0(t))$ is (up to a scaling factor) $(\frac{dy_0(t)}{dt}, -\frac{dx_0(t)}{dt})$.

Linear First Order Equation with 2 independent variables Consider an initial curve C parametrized by $(x_0(t), y_0(t)), t \in I$, $x_0, y_0 \in C^1$ and initial data a function $\phi(t) \in C^1$. Then we want to find $u(x, y)$ defined on $\Omega \subset \mathbb{R}^2$ with $C \subset \Omega$ such that $a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$ and on the curve C, $u(x_0(t), y_0(t)) = \phi(t), t \in I$.

Theorem 5.4.1 Existence and Uniqueness of First Order Equation Let $(x_0, y_0) \in C$ and suppose C is not characteristic there with respect to the PDE $a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$. Then in a neighbourhood U of that point there exists a unique solution of the PDE satisfying the initial condition at every point of $C \cap U$.

Theorem 5.4.2 Initial curve is characteristic but does not match Suppose that the initial curve C is characteristic with respect to the PDE $a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$ at (x_0, y_0) and that:

$$\frac{\frac{d\phi(t_0)}{dt}}{f(x_0 - y_0) - c(x_0, y_0)\phi(t_0)} \neq \mu \equiv \frac{\frac{dx_0(t_0)}{dt}}{a(x_0, y_0)} = \frac{\frac{dy_0(t_0)}{dt}}{b(x_0, y_0)}$$

where the last equality has to hold because of the characteristic requirement. Then there is not solution to the initial value problem in any neighbourhood of the point (x_0, y_0).

Theorem 5.4.3 Initial curve is characteristic and matches ratio Suppose that

$$\frac{\frac{d\phi(t_0)}{dt}}{f(x_0 - y_0) - c(x_0, y_0)\phi(t_0)} = \mu \equiv \frac{\frac{dx_0(t_0)}{dt}}{a(x_0, y_0)} = \frac{\frac{dy_0(t_0)}{dt}}{b(x_0, y_0)}$$

is satisfied for all $t \in I$. Then in a neighbourhood of $(x_0, y_0) = (x_0(t_0), y_0(t_0))$ the initial value problem has infinitely many solutions.

Analytic surface A surface $S \in \mathbb{R}^n$ is analytic if it is a level surface of an analytic function $F(x_1, \ldots, x_n) = 0$ with non-vanishing gradient.

Theorem 5.5.1 Cauchy-Kovalesky Theorem Consider the linear PDE $\sum_{|\alpha| \le m} a^\alpha D^\alpha u = f$. Let x^0 be a point of the initial surface S. Suppose that the coefficients a^α, the RHS f, the initial data $u|_s = \phi_0$, $\frac{\partial u}{\partial n}\big|_s = \phi_1, \ldots, \frac{\partial^{m-1} u}{\partial n^{m-1}}\Big|_s = \phi_{m-1}$ functions $\phi_0, \ldots, \phi_{m-1}$ and the initial surface S are all analytic in some neighbourhood of x^0. Suppose that the initial surface S is not characteristic at x^0 with respect to the PDE, so $\sum_{|\alpha|=m} a^\alpha(x^0)[n(x^0)]^\alpha \neq 0$. Then the Cauchy Problem has a solution $u(x)$ defined and analytic in a neighbourhood of x^0 and this solution is unique in the class of analytic functions.

Theorem 5.5.2 Holmgren's Uniqueness Theorem Suppose that all assumptions of the Cauch-Kovalevsky Theorem hold. Then any two solutions of the Cauchy problem which are defined and are of class C^m in some neighbourhood of x^0 must be equal in some neighbourhood of x^0.

Definition: Canonical Form of General Linear First Order PDE Consider $a(x, y)u_x + b(x, y)u_y + c(x, y)u + d(x, y) = 0$. Then in some new coordinate system (ξ, η), we have $u_\xi + \gamma(\xi, \eta)u + \delta(\xi, \eta) = 0$. The coordinate transform $\eta(x, y)$ must be a solution (which will be non-unique anyway) of the PDE $a\eta_x + b\eta_y = 0$ and $\xi(x, y)$ is subject only to the condition that the transformation Jacobian is non-zero.

Stopped at Page 137 Chapter 5 Classification of 2nd order eqns

ACM95c Book Notes (Ch7-9)

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Definition 7.1.1 Harmonic Function Let $\Omega \subset \mathbb{R}^n$. A function $u \in C^2(\Omega)$ which satisfies Laplace's equation $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$ in Ω is called Harmonic in Ω.

Theorem 7.1.1 Solutions to Laplace Equation are Analytic Let u be a continuous solution of Laplace's equation in Ω. Then u is analytic in Ω.

Laplace Operator in Polar Coordinates (Equation 7.2.2)

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Laplace Operator in Spherical Coordinates (Equation 7.2.3-4)

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

NOTE: ϕ is the polar angle and θ is the azimuthal angle here!

Solutions to Laplace Equation in 2D (Equation 2.19)

$$u_\mu(r,\theta) = R_\mu(r)\Theta_\mu(\theta) = \begin{cases} 1, r^n\cos n\theta, r^n \sin n\theta, & n = 1,2,\ldots \\ \log r, r^{-n}\cos n\theta, r^{-n}\sin n\theta, & n = 1,2,\ldots \end{cases}$$

if Ω does not contain the origin of $\mathbb{R}^2$, all functions above are harmonic in Ω. If it does contain the origin, only functions on the first line are harmonic in Ω.

Solutions to Laplace Equation in 3D (Equation 2.27)

$$u_{n,k}(r,\theta,\phi) = \begin{cases} r^n Y_n^k(\theta,\phi), & k = 1,2,\ldots,2n+1, n = 0,1,2,\ldots \\ r^{-n-1}Y_n^k(\theta,\phi), & k = 1,2,\ldots,2n+1, n = 0,1,2,\ldots \end{cases}$$

If Ω does not contain the origin, all functions above are harmonic in Ω. Otherwise, only the first line functions are harmonic in Ω.

Change of variables] Let Ω and Ω' be two domains in $\mathbb{R}^2$ and let there be a one-to-one mapping from Ω to Ω': $x' = x'(x,y), y' = y'(x,y)$ with inverse mapping $x = x(x',y')y = y(x',y')$. Let the mappings be C^2 in their respective domains. Let $u(x,y)$ be a given function in Ω. Define $u(x',y') = u(x(x',y'), y(x',y'))$. Under the following elementary transformations, $u(x',y')$ satisfies Laplace's equation whenever $u(x,y)$ satisfies Laplace's equation:

- Translations:

$$x' = x + x_0, \quad y' = y + y_0$$
$$x = x' - x_0, \quad y = y' - y_0$$

- Rotations:

$$x' = x\cos\alpha + y\sin\alpha, \quad y' = -x\sin\alpha + y\cos\alpha$$
$$x = x'\cos\alpha - y'\sin\alpha, \quad y = x'\sin\alpha + y'\cos\alpha$$

- Reflections about any straight line in $\mathbb{R}^2$.
- Similarity transformations (i.e. scaling).

Theorem 7.3.1 Linear Transformation that preserves Harmoniticity A linear transformation of coordinates preserves the harmoniticity of every harmonic function iff it is given by a transformation of the form: $\boldsymbol{A} = \lambda \boldsymbol{B}$ where λ is a positive constant and $\boldsymbol{B}$ is an orthogonal matrix. Recall that a matrix is orthogonal if $\sum_{i=1}^n b_{ik}b_{jk} = \delta_{ij}$. Then the transformation is $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x'}$ with inverse $\boldsymbol{x'} = \boldsymbol{A}^{-1}\boldsymbol{x}$.

Inversion with respect to a circle (Equation 7.3.13-15) Consider the boundary of a circle centred at the origin and with radius $a : S(0,a)$. Then the mapping $r^* = \frac{a^2}{r}, \theta^* = \theta$ with inverse $r = \frac{a^2}{r^*}, \theta = \theta^*$ is an inversion with respect to the circle. Then $u(r^*, \theta^*)$ is harmonic whenever $u(r,\theta)$ is harmonic.

Inversion with respect to a sphere (Equation 7.3.16-17)

$$rr^* = a^2, \theta^* = \theta, \phi^* = \phi$$

If $u(r,\theta,\phi)$ is harmonic in Ω, then $\frac{a}{r^*} u\left(\frac{a^2}{r^*}, \theta^*, \phi^*\right)$ is harmonic in Ω^* with respect to r^*, θ^*, ϕ^*.

Harmonic Function from Inversion with respect to circular thing (Equation 7.3.18-22) Let $\boldsymbol{r}, \boldsymbol{r^*}$ be positions vectors a point and its inverse in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Then if $u(\boldsymbol{r})$ is harmonic in Ω, then $u^*(\boldsymbol{r^*}) \equiv \frac{a}{r^*} u\left(\frac{a^2}{(r^*)^2}\boldsymbol{r^*}\right)$ is also harmonic in Ω^*.

Dirichlet Problem or First Boundary Value Problem Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and let f be a given function which is defined and continuous on $\partial\Omega$. We want to find a function u which is defined and continuous on the closure $\bar{\Omega}$ of Ω such that u is harmonic and u is equal to f on $\partial\Omega$. f is the boundary data and $u(x) = f(x), x \in \partial\Omega$ is the boundary condition.

Neumann Problem or Second Boundary Value Problem Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $\boldsymbol{n}(x)$ be the outward unit normal vector to $\partial\Omega$ at the point x. Let f be a given function defined and continuous on $\partial\Omega$. We want to find a function u defined and continuous in $\bar{\Omega}$ such that u is harmonic in Ω and such that the outer normal derivative $\frac{\partial u(x)}{\partial n}$ on $\partial\Omega$ equal to $f(x)$.

Mixed Problem or Third Boundary Value Problem Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $\boldsymbol{n}(x)$ be the outward unit normal vector to $\partial\Omega$ at the point x. Let α, β, f be given functions defined and continuous on $\partial\Omega$. We want to find a function u such that $\Delta u = 0$ in Ω and $\alpha(x)\frac{\partial u(x)}{\partial n} + \beta(x)u(x) = f(x)$ for $x \in \partial\Omega$.

Theorem 7.5.1 Representation theorem for n=3 Let Ω_0 be a bound normal domain in $\mathbb{R}^3$ and let $\boldsymbol{n}$ be the unit exterior normal to the boundary $\partial\Omega_0$. Let u be any function in $C^2(\bar{\Omega}_0)$. Then the value of u at any interior point $\boldsymbol{r}_0 \in \Omega_0$ is:

$$u(\boldsymbol{r}_0) = \frac{1}{4\pi}\int_{\partial\Omega_0}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r_0}|}\frac{\partial u(\boldsymbol{r})}{\partial n} - u(\boldsymbol{r})\frac{\partial}{\partial n}\frac{1}{|\boldsymbol{r}-\boldsymbol{r_0}|}\right]d\sigma - \frac{1}{4\pi}\int_{\Omega_0}\frac{\nabla^2 u(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{r_0}|}dv$$

Note that the last term vanishes if u is a solution to Laplace's equation.

Theorem 7.5.2 Representation theorem for n=2 Let Ω_0 be a bounded normal domain in $\mathbb{R}^2$ and let $\boldsymbol{n}$ be the unit exterior normal to the boundary $\partial\Omega_0$. Let u be any function in $C^2(\bar{\Omega}_0)$. Then the value of u at any point $\boldsymbol{r}_0 \in \Omega_0$ is given by:

$$u(\boldsymbol{r}_0) = \frac{1}{2\pi}\int_{\partial\Omega_0}\left[\log\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\frac{\partial u(\boldsymbol{r})}{\partial n} - u(\boldsymbol{r})\frac{\partial}{\partial n}\log\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\right]ds - \frac{1}{2\pi}\int_{\Omega_0}\nabla^2 u(\boldsymbol{r})\log\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}dxdy$$

Note that the last term vanishes if u is a solution to Laplace's equation.

Definition 7.5.1 Mean Value Property A function $u(\boldsymbol{r})$ defined on $\Omega \subset \mathbb{R}^n$ satisfies the mean value property in Ω if:

$$u(\boldsymbol{r_0}) = \frac{1}{|S(\boldsymbol{r}_0,\delta)|}\int_{S(\boldsymbol{r}_0,\delta)} u(\boldsymbol{r})d\sigma$$

for every $\delta > 0$ such that $\bar{B}(\boldsymbol{r}_0,\delta) \subset \Omega$. Note that $|S(\boldsymbol{r}_0,\delta)|$ is the area of the sphere (or circle) centred at $\boldsymbol{r}_0$ with radius δ.

Theorem 7.5.3 Mean Value Theorem Let u be a harmonic function in a domain $\Omega \subset \mathbb{R}^n$. Then u has the mean value property in Ω.

Theorem 7.5.4 Maximum Principle Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose that u is defined and continuous in $\bar{\Omega}$ and harmonic in Ω. Then u attains its maximum and minimum values on the boundary $\partial\Omega$. If u is not constant on $\bar{\Omega}$, then u attains is maximum and minimum values only on $\partial\Omega$.

Well-posedness (i) There exists a solution, (ii) the solution is unique (iii) the solution depends continuously on the boundary data.

Theorem 7.6.1 Uniqueness of Dirichlet problem The Dirichlet problem has at most one solution.

Theorem 7.6.2 Continuous dependence on data for Dirichlet problem Let f_1 and f_2 be two functions defined and continuous on $\partial\Omega$. Let u_1 be the solution of the Dirichlet problem with $f = f_1$ and let u_2 be the solution of the problem with $f = f_2$. For any $\epsilon > 0$, if $|f_1(x) - f_2(x)| < \epsilon, \forall x \in \partial\Omega$ then $|u_1(x) - u_2(x)| < \epsilon, \forall x \in \bar{\Omega}$.

Theorem 7.6.3 Existence of solution for Dirichlet problem for n=3 Let Ω be a bounded domain in $\mathbb{R}^3$. Then if each point $x \in \partial\Omega$ can be touched by the tip of some probe, which is formed by rotating the curves $y = x^k, x \geq 0, k \geq 1$ about the axis, such that all points of the probe within some positive number ρ from the tip of the probe lie outside Ω, then the Dirichlet problem always has a solution.

Theorem 7.7.1 Formal Solution of Dirichlet Problem on Unit Disk We want to find function $u(r,\theta) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ (that is, twice continuously differentiable on the interior and continuous on the boundary) such that $\nabla^2 u(r,\theta) = 0, 0 \leq r < 1, -\pi \leq \theta \leq \pi$ and $u(1,\theta) = f(\theta), -\pi \leq \theta \leq \pi$ where f is a given function in $C^0(\partial\Omega)$ that satisfies $f(-\pi) = f(\pi)$ and has piecewise continuous derivative. Then the solution $u(r,\theta)$ of the Dirichlet problem is given by the series

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

with coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$$

Theorem 7.7.2 Solution to Dirichlet problem on Unit Disk without piecewise continuous derivative Suppose that $f \in C^0(\partial\Omega)$. Then the solution of the Dirichlet problem on the unit disk is:

$$u(r,\theta) = \begin{cases} \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta), & r < 1 \\ f(\theta), & r = 1 \end{cases}$$

Theorem 7.7.3 Solution to Dirichlet problem using Poisson's integral Suppose that $f \in C^0(\partial\Omega)$. Then the solution of the Dirichlet problem is given by:

$$u(r,\theta) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)f(\phi)}{1+r^2-2r\cos(\theta-\phi)} d\phi, & r < 1 \\ f(\theta), & r = 1 \end{cases}$$

When the disk has radius a, then:

$$u(r,\theta) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2-r^2)f(\phi)}{a^2+r^2-2ar\cos(\theta-\phi)} d\phi, & r < a \\ f(\theta), & r = a \end{cases}$$

and the series solution looks like

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta)$$

Definition 7.8.2 Fourier Series Let f be a piecewise continuous function on $[-\pi, \pi]$. The Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

Theorem 7.8.1 Fourier Theorem Suppose a function f and its derivative are piecewise continuous on $[-\pi, \pi]$ and that f is periodic of period 2π. Then the Fourier series converges to $f(x)$ at every point x where f is continuous and converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ where f has a jump discontinuity.

Parseval Relation (Equation 7.8.34) For every function piecewise continuous on $[-\pi, \pi]$, the Fourier series satisfies:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx$$

Theorem 7.8.2 Convergence of Fourier Series Let f be continuous on the interval $[-\pi, \pi]$, with $f(-\pi) = f(\pi)$. Let the first derivative be piecewise continuous on $[-\pi, \pi]$. Then the Fourier series converges absolutely and uniformly on the interval $[-\pi, \pi]$.

Fourier series over an arbitrary interval (Equation 7.8.37-38) Let $f(x)$ be periodic of period $2L$ that is piecewise continuous on $[-L, L]$. Then the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Fourier cosine and sine series (Equations 7.8.39-40) Let $f(x)$ be defined on $[0, L]$.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$
$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Definition 7.9.1 Green's Function in 3D Let $\Omega \subset \mathbb{R}^3$. Let $h(\boldsymbol{r}', \boldsymbol{r})$ satisfy:

$$\Delta' h(\boldsymbol{r}') = 0, \quad \boldsymbol{r}' \in \Omega$$
$$h(\boldsymbol{r}') = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|}, \quad \boldsymbol{r}' \in \partial\Omega$$

for all $\boldsymbol{r} \in \Omega$. Then the function:

$$G(\boldsymbol{r}', \boldsymbol{r}) = \frac{1}{4\pi}\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} + h(\boldsymbol{r}', \boldsymbol{r}), \quad \boldsymbol{r}', \boldsymbol{r} \in \bar{\Omega}, \boldsymbol{r}' \neq \boldsymbol{r}$$

is called the Green's function for the Dirichlet problem for Ω.

The solution to the Dirichlet problem:

$$\begin{aligned} \Delta u &= 0, \quad \boldsymbol{r} \in \Omega \\ u &= f, \quad \boldsymbol{r} \in \partial\Omega \end{aligned}$$

is hence given by:

$$u(\boldsymbol{r}) = -\int_{\partial\Omega} f(\boldsymbol{r}') \frac{\partial}{\partial n} G(\boldsymbol{r}', \boldsymbol{r}) d\sigma, \quad \boldsymbol{r} \in \Omega$$

Definition 7.9.2 Green's function in 2D Let $\Omega \subset \mathbb{R}^2$. Let $h(\boldsymbol{r}', \boldsymbol{r})$ satisfy:

$$\begin{aligned} \Delta' h(\boldsymbol{r}', \boldsymbol{r}) &= 0, \quad \boldsymbol{r}' \in \Omega \\ h(\boldsymbol{r}', \boldsymbol{r}) &= -\frac{1}{2\pi} \log \frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|}, \quad \boldsymbol{r}' \in \partial\Omega \end{aligned}$$

for all $\boldsymbol{r} \in \Omega$. Then the function:

$$G(\boldsymbol{r}', \boldsymbol{r}) = \frac{1}{2\pi} \log \frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} + h(\boldsymbol{r}', \boldsymbol{r}), \quad \boldsymbol{r}', \boldsymbol{r} \in \bar{\Omega}, \boldsymbol{r}' \neq \boldsymbol{r}$$

is called the Green's function for the Dirichlet problem for Ω. The general solution is given by the same form as the 3D case.

Poisson Kernel (Equation 7.10.13) Consider the Dirichlet problem for $\Omega = B(0, a)$

Leibniz Rule for Differentiating under Integral sign

$$\frac{d}{dy}\left(\int_{a(y)}^{b(y)} f(x, y) dx\right) = \int_{a(y)}^{b(y)} \partial_y f(x, y) dx + f(b(y), y) b'(y) - f(a(y), y) a'(y)$$