Chapter 1

Week 1

ACM95c Class Notes

LIM SOON WEI DANIEL

Quasilinear: Coefficient of the highest order derivative does not depend on the derivative of the unknown function

1.1 Wednesday 1 Apr 2015

4D Method of characteristics Consider the equation:

$$au_x + bu_y + cu_z + = f(x, y, z, u)$$

Write $S = (x, y, z, u(x, y, z))$. Then take the total derivative and set to the source term:

$$\frac{d}{ds}u(x(s), y(s), z(s)) = f(x, y, z, u)$$

Then we have the system of ODEs:

$$\begin{aligned}\dot{x} &= a\\ \dot{y} &= b\\ \dot{z} &= c\\ \dot{u} &= f\end{aligned}$$

Chapter 2

Week 2

2.1 Wednesday 8 Apr 2015

Method of characteristics for 1st order equations Consider $F(\nabla u, u, \vec{x}) = 0$ in $\Omega \subset \mathbb{R}^2$ such that $u = g$ on $\Gamma \subseteq \partial\Omega$. We require $F, g \in C^1$. If F is not fully non-linear, then we can write it as the quasi-linear form:

$$F = a(x, y, u)u_x + b(x, y, u)u_y + c(x, y, u)$$

Then we pick a curve C in Ω that starts at some point $x_0 \in \Gamma$ on the initial data. Parametrize this curve using $s \in \mathbb{R}$: $\vec{x}(s) = (x(s), y(s))$. Also define $z(s) = u(\vec{x}(s))$, the value of u along the curve C. Define $\vec{p}(s) = \nabla u(\vec{X}(s))$, the gradient of u along the curve C. Note that we chose p because it corresponds to the classical momentum. Now the problem is as follows: We want to find $\vec{X}(s)$ such that $z(s), p(s)$ are easy to solve from the system.

Note that the ith component of p is:

$$p_i(s) = u_{x_i}(\vec{X}(s))$$

Taking the derivative with respect to s,

$$\dot{p}_i(s) = \sum_{j=1}^{n} u_{x_i,x_j}(\vec{x}(s))\dot{x}_j(s)$$

Now we want an ODE that only depends on $\vec{x}, z$ and p alone to relate these coordinates. Hence we want to get rid of u_{x_i,x_j}. Returning to the PDE, we can take the derivative with respect to x_i:

$$0 = F(\nabla u, u, \vec{x})$$

$$\implies 0 = \frac{\partial}{\partial x_i} F(\nabla u, u, \vec{x}) = \sum_{j=1}^{n} \frac{\partial F(\nabla u, u, \vec{x})}{\partial p_j} u_{x_j,x_i} + \frac{\partial F(\nabla u, u, \vec{x})}{\partial z} u_{x_i} + \frac{\partial F(\nabla u, u, \vec{x})}{\partial x_i}$$

Along the characteristic curve C, $u_{x_i}(s) = p_i(s)$.

$$0 = \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), \vec{x}(s))}{\partial p_j} u_{x_i,x_j} + \frac{\partial F(p(s), z(s), \vec{x}(s))}{\partial z} p_i(s) + \frac{\partial F(p(s), z(s), \vec{x}(s))}{\partial x_i} \tag{2.1}$$

Comparing this equation to $\dot{p}_i(s) = \sum_{j=1}^{n} u_{x,x_j}(\vec{x}(s))\dot{x}_j(s)$, we note that if we require the term with second derivatives to equal to the equation for $\dot{p}_i$:

$$\dot{p}_i(s) = \sum_{j=1}^{n} u_{x_i,x_j}\dot{x}_j(s) = \sum_{j=1}^{n} u_{x_i,x_j}\frac{\partial F}{\partial p_j} \implies \dot{x}_j = \frac{\partial F}{\partial p_j}$$

after comparing coefficients. Under this assumption, we replace the first term in (2.1) to have the characteristic equation (shifting $\dot{p}_i$ to the LHS):

$$\dot{p}_i = -\frac{\partial F}{\partial z}p_i - \frac{\partial F}{\partial x_i}$$

Hence we have the system:

$$\vec{x} = \nabla_{\vec{p}}F, \quad \dot{\vec{p}} = -(\nabla_z F)\vec{p} - \nabla_x F$$

We now examine the derivative of $z(s)$:

$$\dot{z}(s) = (u(\vec{x}(s)))' = \sum_{j=1}^{n} \frac{\partial u(\vec{x}(s))}{\partial x_j}\dot{x}_j(s) = \vec{p}\cdot\nabla_p F$$

after substituting expressions for $u_{x_j} = p_j$ and $\dot{x}_j = (\nabla_p F)_j = \frac{\partial F}{\partial p_j}$. This final equation gives us the complete closed system:

$$\dot{\vec{x}} = \nabla_p F, \quad \text{n components} \tag{2.2}$$
$$\dot{z} = \vec{p}\cdot\nabla_p F, \quad \text{1 component} \tag{2.3}$$
$$\dot{\vec{p}} = -\nabla_x F - (\nabla_z F)\vec{p}, \quad \text{n components,} \tag{2.4}$$

Hence this ODE system has $2n+1$ components.

Theorem Let $u \in C^2(\Omega)$ and $F(\nabla u, u, \vec{x}) = 0$. If $\vec{x}(s)$ solves (2.2), then $\vec{p} = \nabla u(\vec{x}(s))$ and $z(s) = u(\vec{x}(s))$ solves (2.3) and (2.4) respectively for all $\vec{x}(s) \in \Omega$.

2D Linear Case Write $F(u_x, u_y, u, x, y) = a(x,y)u_x + b(x,y)u_y + c(x,y)u = a(x,y)p_1 + b(x,y)p_2 + c(x,y)z = 0$. Then we have the system of ODEs:

$$\dot{x} = \frac{\partial F}{\partial p_1} = a(x,y)$$
$$\dot{y} = \frac{\partial F}{\partial p_2} = b(x,y)$$
$$\dot{z} = \nabla_p F\cdot\vec{p} = a(x,y)p_1 + b(x,y)p_2 = -c(x,y)z$$

2D Quasilinear Case Let $F(u_x, u_y, u, x, y) = a(x,y,u)u_x + b(x,y,u)u_y + c(x,y,u) = 0$. Then we write these using momentum variables along the characteristic as:

$$F(p_1, p_2, z, x, y) = a(x,y,z)p_1 + b(x,y,z)p_2 + c(x,y,z)$$

which gives us the ODE system:

$$\dot{x} = a(x,y,z)$$
$$\dot{y} = b(x,y,z)$$
$$\dot{z} = \nabla_p F\cdot\vec{p} = ap_1 + bp_2 = -c(x,y,z)$$

Example 1 Consider $u_x u_y = u(x,y)$ in $\{x > 0\}$ with initial data $u(0,y) = y^2$. We want to transform the PDE into a form so that one side is zero. This gives us $F(u_x, u_y, u) = u_x u_y - u = 0$. Then we have the corresponding characteristic form:

$$F(p_1, p_2, z) = p_1 p_2 - z$$

Applying the general method of characteristics to obtain an ODE system:

$$\begin{aligned} \dot{x} &= p_2 \\ \dot{y} &= p_1 \\ \dot{z} &= \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} p_2 \\ p_1 \end{pmatrix} = 2p_1 p_2 = 2z \end{aligned}$$

We now examine the expression for the derivatives of p from equation (2.4):

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial F}{\partial x} - \frac{\partial F}{\partial z} p_1 = -p_1 \cdot -1 = p_1 \\ \dot{p}_2 &= -\frac{\partial F}{\partial y} - \frac{\partial F}{\partial z} p_2 = -1 \cdot (-p_2) = p_2 \end{aligned}$$

because F has no dependence on x or y. The initial conditions are:

$$\begin{aligned} x(0) &= 0 \\ y(0) &= y_0 \\ z(0) &= u(x(0), y(0)) = u(0, y_0) = y_0^2 \end{aligned}$$

which gives us the initial conditions for the momentum variables:

$$\begin{aligned} p_1(0) &= u_x(0, y_0) \\ p_2(0) &= u_y(0, y_0) = 2y_0 \end{aligned}$$

and to obtain $p_1(0)$ we use the characteristic equation $p_1(0)p_2(0) = z(0) \implies p_1(0) = \frac{z(0)}{p_2(0)} = \frac{y_0^2}{2y_0} = \frac{y_0}{2}$. Solving the ODEs for p_1 and p_2:

$$\begin{aligned} \dot{p}_1 = p_1, p_1(0) = \frac{y_0}{2} &\implies p_1(s) = \frac{y_0}{2} e^s \\ \dot{p}_2 = p_2, p_2(0) = 2y_0 &\implies p_2(s) = 2y_0 e^s \\ \dot{z} = 2z, z(0) = y_0^2 &\implies z(s) = y_0^2 e^{2s} \end{aligned}$$

Now we have the expressions for x, y:

$$\begin{aligned} \dot{x} = p_2(s) = 2y_0 e^s, x(0) = 0 &\implies x(s) = 2y_0 e^s - 2y_0 \\ \dot{y} = p_1(s) = \frac{y_0}{2} e^s, y(0) = y_0 &\implies y(s) = \frac{y_0}{2} e^s + \frac{y_0}{2} \end{aligned}$$

We now can invert the system of equations to express y_0, e^s in terms of x, y.

$$\begin{aligned} y_0 &= \frac{4y - x}{4} \\ e^s &= \frac{2x + 4y - x}{4y - x} \end{aligned}$$

which we can plug back into the expression for $u(x,y)$ to obtain:

$$u(x,y) = z = y_0^2 e^{2s} = \frac{(4y+x)^2}{16}$$

which is the solution we want.

2.2 Recitation Wednesday 8 Apr 2015

Example 1 Consider $u_t + u_x = 0, u(x,0) = f(x), -\infty < x < \infty$. We want to write the LHS of the PDE in the form $\frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = \frac{du}{ds} = 0$. For this to be true, we would like $\frac{dt}{ds} = 1, \frac{dx}{ds} = 1$. We pick the initial condition $t(s=0) = 0$. Then $u(x(0), t(0)) = u(x(0), 0) = f(x(0))$. Hence we define $x_0 = x(0)$ so we have the initial condition for $u(0) = f(x_0)$. Solving, $t(s) = s, x(s) = s + x_0, u(s) = f(x_0)$. Note that $(x(s), t(s))$ describes a curve on which $u(s)$ is constant. Call this a characteristic curve. We note that we can describe the curve explicitly by eliminating s: this gives us $x - t = x_0$, which is a family of curves indexed by x_0. Since the value of u on each of the characteristics is a constant, we have that $u(x,t) = u(x_0, 0) = f(x_0) = f(x-t)$.

Example 2 $u_x - 2xu_t = 0, -\infty < x < \infty, u(x,0) = f(x)$. We hence use the method of characteristics to write the ODE system:

$$\begin{aligned}\dot{x}(s) &= 1\\ \dot{t}(s) &= -2x\\ \dot{u}(s) &= 0\end{aligned}$$

With initial conditions:

$$\begin{aligned}x(0) &= x_0\\ t(0) &= 0\\ u(0) &= f(x_0)\end{aligned}$$

and hence we solve the system:

$$\begin{aligned}u(s) &= f(x_0)\\ x(s) &= s + x_0\\ \dot{t}(s) = -2(s + x_0) &\implies t(s) = -s^2 - 2sx_0\end{aligned}$$

Combining the expressions for $x(s), t(s)$, we obtain:

$$t = -(x - x_0)^2 - 2(x - x_0)x_0 \implies t = -x^2 + x_0^2$$

Hence the characteristics are inverted parabolas with x intercept equal to x_0^2. But x_0^2 is non-negative. This means that the characteristics will cut the x axis twice (if $x_0^2 > 0$)! But the x-axis is precisely where we have defined the initial data. Hence we have the requirement that $f(x) = f(-x)$, and that $f(x)$ has to be even so that the value of u on the characteristics is a constant. If f is not even then we do not have a solution. Hence $f(x)$ has to be even for there to be a solution.

2.3 Friday 10 Apr 2015

Example 2 $|\nabla u|^2 = u_x^2 + u_y^2 = 1$ with initial data $u(x,y) = 1, x^2 + y^2 = 1$. This is a fully non-linear problem. We recall the conditions that the characteristic functions have to fulfil:

$$
\begin{aligned}
F(\nabla u, u(\vec{x})) = 0 \iff F(\vec{p}(s), z(s), \vec{x}(s)) = 0 \\
\dot{\vec{x}} = \nabla_{\vec{p}} F \\
\dot{z} = \nabla_{\vec{p}} F \cdot \vec{p} \\
\dot{\vec{p}} = -\nabla_x F - \vec{p}\frac{\partial F}{\partial z}
\end{aligned}
$$

Then we define the function:

$$F(p_1, p_2) = p_1^2 + p_2^2 - 1 = 0$$

Then we have the system:

$$
\begin{aligned}
\dot{x} = 2p_1 \\
\dot{y} = 2p_2 \\
\dot{z} = 2p_1p_1 + 2p_2p_2 = 2(p_1^2 + p_2^2) = 2
\end{aligned}
$$

Now we note that F has no dependence on $\vec{x}$ or z. Hence we have:

$$
\begin{aligned}
\dot{p}_1 = 0 \implies p_1(s) = p_1(0) \\
\dot{p}_2 = 0 \implies p_2(s) = p_2(0)
\end{aligned}
$$

and the values of p_1, p_2, which are the values of the gradient of u in the x and y direction respectively along the characteristic curve, do not change along the characteristic.

For the boundary conditions, we parametrise the circle:

$$
\begin{aligned}
x(0) \equiv x_0 \\
y(0) \equiv y_0 \\
x_0^2 + y_0^2 = 1
\end{aligned}
$$

for a certain point (x_0, y_0) on the unit circle. Hence $z(x_0, y_0) = 1$, which is the value of u along the characteristic starting at (x_0, y_0). Now we know that the derivative of u is constant along the characteristic:

$$\frac{\partial u}{\partial T} = \nabla u \cdot T = 0$$

where T is the tangent vector of the characteristic. Now along the curve Γ, which is the initial curve i.e. the unit circle, we write the curve as a level set:

$$\Gamma : x^2 + y^2 - 1 = 0$$

and hence we have the normal to the curve, which is proportional to the gradient:

$$N = (2x, 2y)$$

But we know that the tangent vector is orthogonal to the normal vector hence we note that the vector:

$$T = (2y, -2x)$$

will be the tangent vector. Hence at the point (x_0, y_0) we have the tangent vector:

$$T_0 = (y_0, -x_0)$$

and hence the condition on u at the boundary is:

$$\nabla u \cdot T = 0 \implies (p_1(0), p_2(0)) = (u_x(x_0, y_0), u_y(x_0, y_0)) \cdot (y_0, -x_0) = 0$$
$$\implies p_1(0) = p_2(0)\frac{x_0}{y_0}$$

We still require one more equation to solve for $p_1(0)$ and $p_2(0)$. We return to the PDE. We hence have that at (x_0, y_0):

$$p_1(0)^2 + p_2(0)^2 = 1$$

and we have two conditions for the gradient at the boundary. Solving the equations for $p_1(0), p_2(0)$ simultaneously,

$$p_2(0) = \frac{y_0}{\sqrt{x_0^2 + y_0^2}} = y_0$$
$$p_1(0) = p_2(0)\frac{x_0}{y_0} = x_0$$

Hence we know $p_1(s) = p_1(s) = y_0$ and $p_2(s) = p_2(0) = x_0$. If we take the negative solution to the square root, we can get a different solution.

Now the equations on $z(s)$ are:

$$\dot{z} = 2, z(0) = 1 \implies z = 2s + 1$$

Now we have p_1, p_2, z. We need to solve for the projected characteristics x, y. We have the ODE system:

$$\dot{x} = 2p_1(s) = 2x_0, x(0) = x_0 \implies x(s) = 2x_0 s + x_0 = x_0(2s + 1)$$
$$\dot{y} = 2p_2(s) = 2y_0, y(0) = y_0 \implies y(s) = y_0(2s + 1)$$

We now have x and y in terms of x_0, y_0 and s. We need to invert the system. Now we know that the squares of x_0 and y_0 sum to unity hence we square the equations and add them:

$$x(s)^2 + y(s)^2 = (x_0^2 + y_0^2)(2s + 1)^2 = (2s + 1)^2$$

We have successfully eliminated x_0, y_0 by considering the initial curve which constrained x_0 and y_0. Solving for s, we have:

$$s = \frac{\sqrt{x^2 + y^2} - 1}{2}$$

Now we know that z represents the value of u along the characteristic, and it had the value $2s + 1$. Hence we have, substituting the expression for s into the expression for z to obtain:

$$z(s) = 2s + 1 \implies u(x, y) = \sqrt{x^2 + y^2}$$

Note that if we had taken the negative sign for the square root just now, we would have obtained the solution:

$$u(x, y) = 2 - \sqrt{x^2 + y^2}$$

Note that the functions are not differentiable at $(x, y) = (0, 0)$. Hence the solution does not satisfy the PDE everywhere.

Summary: Take PDE, write in characteristic form, solve as many ODEs (the easy ones first), then solve for the auxiliary variables x_0, y_0, s, plug into z, and obtain the solution.

Generally If we have an n-dimensional problem $u(x_1, x_2, \ldots, x_n = 0) = 0$ (the last variable means that we have data along that axis $x_n = 0$, just like knowing $u(x, 0)$ for the $u(x, y)$ problem), then the boundary data can be written in the form:

$$p_i(\vec{x}(0)) = g_{x_i}(\vec{x}(0)), i = 1, 2, \ldots, n-1$$
$$F(\vec{p}(0), z(0), \vec{x}(0)) = 0$$

where g_{x_i} are known functions.

Conditions on the initial data We require that the characteristics are not tangential to the initial data. This is equivalent to saying that:

$$\dot{\vec{x}} \cdot N(\vec{x}(0)) \neq 0 \iff \nabla_{\vec{p}} F \cdot N(\vec{x}(0) \neq 0 \quad \text{to check}$$

Hamilton-Jacobi Equations $u_t + H(\nabla u) = 0, u(x, 0) = g(x)$. Consider the specific example $u_t + \frac{u_x^2}{2} = 0, u(x, 0) = x^2$. We write it in characteristic variables:

$$p_1 = u_t(\vec{x}(s), t(s))$$
$$p_2 = u_x(\vec{x}(s), t(s))$$

with the function:

$$F(p_1, p_2) = p_1 + \frac{p_2^2}{2} = 0$$

Then we require the following ODEs to be solved:

$$\dot{t} = 1 \quad \text{the coefficient of } u_t$$
$$\dot{x} = p_2(s)$$
$$\dot{z} = \vec{p} \cdot \nabla_{\vec{p}} F = p_1 + p_2^2$$
$$\dot{p}_1 = 0$$
$$\dot{p}_2 = 0$$

with initial conditions:

$$t(0) = 0$$
$$x(0) = x_0$$
$$z(0) = x_0^2$$
$$p_2(0) = u_x(x_0, 0) = \frac{d}{dx}(u(x, 0))\Big|_{x=x_0} = 2x_0$$
$$p_1(0) = \frac{-p_2^2(0)}{2} = -2x_0^2 \quad \text{from the PDE}$$

We may immediately solve for t:

$$t(s) = s$$

and we can replace all instances of s by t. Now p_1 and p_2 are constant along the characteristics, referring back to the ODEs. Hence we have:

$$p_1(t) = p_1(0) = -2x_0^2$$
$$p_2(t) = p_2(0) = 2x_0$$

Then we want to solve the ODEs:

$$\dot{x} = p_2(t) = 2x_0, x(0) = x_0 \implies x(t) = x_0(2t+1)$$
$$\implies x_0 = \frac{x}{2t+1}$$

Now we examine z:

$$\dot{z} = p_1 + p_2^2 = -2x_0^2 + (2x_0)^2 = 2x_0^2, z(0) = x_0^2 \implies z(t) = x_0^2(2t+1)$$

Hence the solution is (replacing x_0 using the expression for x and t):

$$u(x,t) = \frac{x^2}{(2t+1)^2}(2t+1) = \frac{x^2}{2t+1}$$

Alternative method for Hamilton-Jacobi Equations Recall the problem:

$$u_t + \frac{u_x^2}{2} = 0, u(x,0) = x^2$$

Now we differentiate the equation once in x:

$$u_{tx} + u_x u_{xx} = 0, u_x(x,0) = 2x$$

and we define:

$$w(x,t) = u_x(x,t)$$

which gives us the quasi-linear system (exchanging the order of second derivatives):

$$w_t + ww_x = 0, w(x,0) = 2x$$

which is precisely Burger's equation. Note that his method of differentiating to get a quasilinear equation only works for Hamilton-Jacobi equations.

Now we solve the quasilinear equation:

$$\dot{t} = 1, t(0) = 0 \implies t(s) = s$$
$$\dot{x} = z, x(0) = x_0$$
$$\dot{z} = 0, z(0) = 2x_0$$

The third equation gives:

$$z(s) = 2x_0$$

so the ODE for x is

$$\dot{x} = 2x_0, x(0) = x_0 \implies x(s) = 2x_0 s + x_0$$

Inverting the equation to solve for x_0,

$$x_0 = \frac{x}{2s+1} = \frac{x}{2t+1}$$

Hence we have:

$$z(s) = 2x_0 \implies w(x,t) = \frac{2x}{2t+1}$$

But we earlier defined that:

$$w = u_x \implies u(x,t) = \int \frac{2x}{2t+1} dx = \frac{x^2}{2t+1} + f(t), f(t) = 0$$

Homogeneous vs Inhomogeneous If x, y appears in the PDE at all.

Linear Examine part that depends on u. Can you write it as $L(c_1 u + c_2 v) = c_1 L(u) + c_2 L(v)$?

Semilinear Highest order term is linear.

Quasilinear Fix all lower order terms. Highest order term is now linear.

Well-posed Unique solution exists which depends continuously on initial data.

Semi-linear problem Parametrize initial curve by q. Ensure problem is non characteristic (Jacobian is non-vanishing at the initial curve). Solve ODEs to obtain $x(s,q)$ and $y(s,q)$. Invert to find $s(x,y)$ and $q(x,y)$. Put into $u(s,q)$ to obtain $u(x,y)$.

Shock: Rankine-Hugoniot Condition $\dot{\sigma} = \frac{F(u_l)-F(u_r)}{u_l-u_r}$, the speed of the shock. For initial conditions, note where the shock starts.

Entropy Condition For the conservation law $u_t + F(u)_x = 0$, $F'(u_l) > \dot{\sigma} > F'(u_r)$.

Decay Rate If u solves a conservation law with flux F that is smooth and uniformly convex ($F'' \geq c > 0$) and $F(0) = 0$, and the initial data g is integrable and bounded, then:

$$|u(x,t)| \leq \frac{C}{\sqrt{t}}$$

where $C > 0$ is a constant.

Fully Non-linear Problem Given $F(\nabla_{\vec{x}} u, u, \vec{x}) = 0$ and $u(\vec{x}) = g(\vec{x})$ on $\vec{x} \in \Gamma$:

$$\begin{aligned} F(\nabla u, u(\vec{x}), \vec{x}) = 0 &\iff F(\vec{p}(s), z(s), \vec{x}(s)) = 0 \\ \dot{\vec{x}} &= \nabla_{\vec{p}} F \\ \dot{z} &= \nabla_{\vec{p}} F \cdot \vec{p} \\ \dot{\vec{p}} &= -\nabla_{\vec{x}} F - \vec{p}\frac{\partial F}{\partial z} \end{aligned}$$

2D Fully Non-linear problem Given $F(u_x, u_y, u, x, y) = 0$ and $u(x,y) = g(x,y), (x,y) \in \Gamma$:

Define:

$$\begin{aligned} p_1(s) &= u_x(x(s), y(s)) \\ p_2(s) &= u_y(x(s), y(s)) \\ z(s) &= u(x(s), y(s)) \\ (x_0, y_0) \in \Gamma &\text{ such that } x(0) = x_0, y(0) = y_0 \\ z(0) &= u(x_0, y_0) \end{aligned}$$

Define $F(p_1, p_2, z, x, y)$ by making the replacements $u_x \to p_1, u_y \to p_2, u \to z$). Let Γ be written as a level set $h(x,y) = 0$, then the normal is $\nabla h = (h_x, h_y)$ so the tangent is $T = (-h_y, h_x)$. Then the directional derivative in the tangent is known because $g(x,y)$ is defined in that direction:

$$(p_1(0), p_2(0)) \cdot (-h_y, h_x) = \nabla g(x_0, y_0) \cdot (-h_y, h_x)$$

and use $\Gamma(x_0, y_0) = 0$ for the second relation to solve for $p_1(0)$ and $p_2(0)$ in terms of x_0 and y_0.

Then the ODE system is:

$$\begin{aligned} \dot{x}(s) &= \frac{\partial F}{\partial p_1}, \quad x(0) = x_0 \\ \dot{y}(s) &= \frac{\partial F}{\partial p_2}, \quad y(0) = y_0 \\ \dot{z} &= p_1 \frac{\partial F}{\partial p_1} + p_2 \frac{\partial F}{\partial p_2} \\ \dot{p}_1 &= -\frac{\partial F}{\partial x} - p_1 \frac{\partial F}{\partial z} \\ \dot{p}_2 &= -\frac{\partial F}{\partial y} - p_2 \frac{\partial F}{\partial z} \end{aligned}$$

Leibniz Rule for Differentiating under Integral sign

$$\begin{aligned} \frac{d}{dy}\left(\int_{a(y)}^{b(y)} f(x,y)dx\right) &= \int_{a(y)}^{b(y)} \partial_y f(x,y)dx \\ &+ f(b(y), y)b'(y) - f(a(y), y)a'(y) \end{aligned}$$

Chapter 3

Week 3

3.1 Monday 13 Apr 2015

Traveling waves Consider $u(x,t) = v(x - at), a \in \mathbb{R}$. Then a represents the speed and v represents the profile of the wave. In n-dimensions,

$$u(\vec{x}, t) = v(\vec{b} \cdot \vec{x} - at)$$

with speed $\frac{a}{|\vec{b}|}$. Call $\vec{b}$ the wavefront. We expect that if we substitute this ansatz into the PDE, we will get an ODE.

Transport equation $u_t + cu_x = 0$. Substitute $u(x,t) = v(x - at) = v(s)$ where $s = x - at$. Then we have the ODE:

$$\begin{aligned} u_t &= -av'(s) \\ u_x &= v'(s)a \\ \implies (-a + c)v' &= 0 \end{aligned}$$

hence we have two cases: $c = a$ and $c \neq a$. Hence $v' = 0$ is a solution, which gives us $v(s) = v(0)$. Hence $u(x,t) = v(x - at)$.

Transport equation with diffusion Consider $u_t + cu_x = \epsilon u_{xx}$. Apply the same technique by substituting the wave solution to obtain an ODE:

$$\begin{aligned} u_t &= -av' \\ u_x &= v' \\ u_{xx} &= v'' \\ (-a + c)v' &= \epsilon v'' \end{aligned}$$

If $\epsilon \neq 0$ and $a = c$, then we require $v'' = 0$ which gives us $v(s) = c_1 s + c_0$. The travelling wave solution in this case is $u(x,t) = c_1(x - at) + c_0$. Although this satisfies the equation, the solution is unbounded.

More generally, when $a \neq c$ and $\epsilon \neq 0$ then we require $v'' = \frac{c-a}{\epsilon} v'$ which immediately has solutions: $v(s) = c_0 e^{\frac{c-a}{\epsilon} s} + c_1$. Note that depending on $c - a$, the solution may be either bounded or unbounded. The solution in this case is: $u(x,t) = c_0 e^{(x-at)\frac{c-a}{\epsilon}} + c_1$.

Burgers' equation $u_t + F(u)_x = 0$. In the inviscid case, $F(u) = \frac{u^2}{2}$. We can consider the viscous version by considering $F_\epsilon(u) = \frac{u^2}{2} - \epsilon u_x$. Then we have the PDE: $u_t + uu_x = \epsilon u_{xx}$. But since the equation has a second derivative, we require that the second derivative exists too, and hence cannot allow shock solutions. Substituting the travelling wave form $u(x,t) = v(x - at)$, we obtain the ODE:

$$-av' + vv' = \epsilon v'' \implies \frac{d^2 v}{ds^2} = \frac{1}{\epsilon}\frac{d}{ds}\left(\frac{v^2}{2} - av\right)$$

This allows us to integrate once in s:

$$\frac{dv}{ds} = \frac{1}{\epsilon}\left(\frac{v^2}{2} - av + \tilde{c}_0\right) = \frac{1}{2\epsilon}(v^2 - 2av + c_0) = \frac{1}{2\epsilon}(v - v_1)(v - v_2)$$

where:

$$v_{1,2} = a \pm \sqrt{a^2 - c_0}$$

Now we make the assumption that $a^2 > c_0 > 0$ so that we have real distinct v_1, v_2. Assume that $v_1 < v_2$. Now we have three cases:

$$\begin{aligned} v = v_1 \text{ or } v_2 &\implies \frac{dv}{ds} = 0 \\ v_1 < v < v_2 &\implies \frac{dv}{ds} < 0 \\ v > v_2, v < v_1 &\implies \frac{dv}{ds} > 0 \end{aligned}$$

Clearly, we can only obtain a travelling wave that is bounded above and below if $v_1 < v < v_2$. Note that v_1 is a stable point and v_2 is an unstable point. Hence we expect $v(s) \to v_1, s \to \infty$ and $v(s) \to v_2, s \to -\infty$.

We now solve the autonomous differential equation explicitly,

$$\begin{aligned} \int \frac{dv}{(v - v_1)(v - v_2)} &= \frac{1}{2\epsilon}\int ds \\ \implies \int dv \frac{1}{v_1 - v_2}\left(\frac{1}{v - v_1} - \frac{1}{v - v_2}\right) &= \frac{s}{2\epsilon} + c \\ \implies \frac{1}{v_1 - v_2}\ln\frac{v - v_1}{v_2 - v} &= \frac{s}{2\epsilon} \end{aligned}$$

where $c = 0$ because we know the bounds at $v \to \pm\infty$. Define the parameter $\beta = \frac{v_2 - v_1}{2\epsilon} > 0$. Then we have the solution:

$$\begin{aligned} v(s) &= \frac{v_1 + e^{-\beta s} v_2}{1 + e^{-\beta s}} \\ v(0) &= \frac{v_1 + v_2}{2} \\ \implies u(x,t) = v(x - at) &= \frac{v_1 + e^{-\beta(x - at)} v_2}{1 + e^{-\beta(x - at)}} \end{aligned}$$

observe that in the limit as $\epsilon \to 0$, we will obtain a shock solution (since we obtain the Burgers' equation without the RHS term). Explicitly, as $\epsilon \to 0, \beta \to \infty$ and hence :

$$v(s) = \begin{cases} v_2, s < 0 \\ \frac{v_1 + v_2}{2}, s = 0 \\ v_1, s > 0 \end{cases}$$

This is virtually identical to the shock solution, with the exception that we have defined the value of a point at $s = 0$.

Exponential solution $u(x,t) = e^{i(k\cdot x - \omega t)}$.

Exponential solution in the transpose equation $u_t + u_x = u_{xx}$. Note that since the exponential is complex, for a linear operator, we will obtain two solutions, one for the real part and one for the imaginary.

Applying the exponential ansatz to the PDE:

$$-i\omega u + iku = -k^2 u \implies (-i\omega + ik + k^2)u = 0$$

If $(-i\omega + ik + k^2) = 0$ then the PDE is satisfied. We write:

$$\omega(k) = -ik^2 + k$$

and substitute this to obtain:

$$u(x,t) = e^{ik(x-t)}e^{-k^2 t}$$

The former $e^{ik(x-t)}$ is the wave part, and the latter $e^{-k^2 t}$ is the decay part. We can take the real and imaginary parts to obtain two solutions:

$$\Re(u) = e^{-k^2 t}\cos[k(x-t)]$$
$$\Im(u) = e^{-k^2 t}\sin[k(x-t)]$$

Application to Quantum Mechanics $iu_t + u_{xx} = 0$. Then we have the ODE:

$$i(-i\omega) - k^2)u = 0$$

with the dispersion relation $\omega = k^2$ and hence solutions of the form:

$$u(x,t) = e^{i(kx - k^2 t)}$$

3.2 Wednesday 15 Apr 2015

Second order linear problems Define the Laplacian in $\mathbb{R}^n : \Delta u = \nabla \cdot \nabla u = \sum_{j=1}^{n} u_{x_j x_j}$

$$\begin{aligned} \Delta u = 0, &\quad \text{Laplace's Equation, Elliptic} \\ u_t = \Delta u, &\quad \text{Heat Equation, Parabolic} \\ u_{tt} = \Delta u, &\quad \text{Wave Equation, Hyperbolic} \\ \Delta u = f &\quad \text{Poisson's Equation} \end{aligned}$$

Derivation of Laplace's Equation Consider u to be a temperature of a region W with F as a heat flux. Consider steady state. Consider the boundary ∂. Then the net flux into the system is zero:

$$\int_{\partial W} \vec{F} \cdot \vec{N} d\sigma = 0$$

Now we apply the Divergence Theorem:

$$\int_{\partial W} \vec{F} \cdot \vec{N} d\sigma = \int_W \nabla \cdot \vec{F} dV = 0$$

Now this identity must hold for all small regions in W. This can only happen if $\nabla \cdot \vec{F} = 0$ everywhere. Now this means that the flux can be written as the gradient of some function:

$$\vec{F} = -c\nabla u, c \neq 0 \implies \nabla \cdot (-c\nabla u) = 0 \implies \nabla^2 u = 0$$

Harmonic Function If $u \in C^2(\Omega)$ and satisfies $\Delta u = 0$ in Ω, then is u is harmonic in Ω.

Theorem: Harmonic functions are analytic If u is continuous and $\Delta u = 0$ in Ω, then u is real analytic in Ω. Real analytic means that $u \in C^\infty(\Omega)$ and it has a convergent Taylor series at every point in Ω.

Making more harmonic functions Translations, rotations, reflections (against lines), scalings of harmonic functions are still harmonic.

Example 2 Given that $u(x, y)$ is harmonic, we consider the second function $v(x, y) = u(\tilde{x}(x, y), \tilde{y}(x, y))$ such that the transformations $\tilde{x}, \tilde{y}$ are a combination of simultaneous translations and rotations etc, then $v(x, y)$ is also harmonic.

Example 3: Undetermined coefficients Suppose we want a harmonic function that is a second order polynomial. Then write: $u(x, y) = ax^2 + by^2 + L.O.T.$ so that the Laplacian is $\Delta u = 2a + 2b$. For this function to be harmonic, we hence require that $b = -a$ and $u(x, y) = a(x^2 - y^2) + L.O.T.$.

Example 4: Higher Order Polynomial Consider the nth order polynomial (just nth order, can be generalised using linearity later):

$$u(x, y) = a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n$$

$$u_{xx} = n(n-1)a_0 x^{n-2} a_1 (n-1)(n-2) y x^{n-3} + \ldots + 2 \cdot 1 \cdot y^{n-2} = \sum_{j=0}^{n-2} a_j (n-j)(n-j-1) y^j x^{n-j-2}$$

$$u_{yy} = 2 \cdot 1 \cdot x^{n-2} + \ldots + a_{n-1}(n-1)(n-2) x y^{n-3} + n(n-1) a_n y^{n-2} = \sum_{j=0}^{n-2} a_{j+2} (j+2)(j+1) x^{n-2-j} y^j$$

and hence for the Laplacian to vanish,

$$\sum_{j=0}^{n-2} a_j (n-j)(n-j-1) y^j x^{n-j-2} + a_{j+2}(j+2)(j+1) x^{n-2-j} y^j = 0$$

$$\implies \sum_{j=0}^{n-2} a_j (n-j)(n-j-1) + a_{j+2}(j+2)(j+1) = 0$$

$$\implies a_{j+2} = -\frac{(n-j)(n-j-1)}{(j+2)(j+1)} a_j$$

and we have a recursion relation. We require initial conditions a_0, a_1 to determine all the other coefficients.

Well-posedness Consider the initial data problem: $\Delta_{x,y} u_1 = 0, u_1(x, 0) = f(x), \frac{\partial u_1(x,0)}{\partial y} = g(x)$. We will like to check if the problem is well-posed. Note that this equation is not well posed, because we can give $f(x) = 0 = g(x)$ and the solution should be zero everywhere. We can perturb this problem by considering the other problem: $\Delta u_2 = 0, u_2(x, 0) = 0, \frac{\partial u_2(x,0)}{\partial y} = \frac{\sin nx}{n}$. Now we can pick n large to that the perturbation is small. The solution to the second problem is given by:

$$u_2(x, y) = \frac{1}{n^2} \sin nx \sinh ny$$

Now consider the difference between the initial data of the two problems:

$$|u_1(x,0) - u_2(x,0)| = 0$$
$$\left|\frac{\partial u_1(x,0)}{\partial y} - \frac{\partial u_2(x,0)}{\partial y}\right| = \frac{1}{n}|\sin(nx)|$$

and the second equation can be made arbitrarily small by appropriate choice of n. However, the solution difference is given by:

$$|u_1(x,y) - u_2(x,y)| = \left|0 - \frac{1}{n^2}\sin nx \sinh ny\right|$$

which has to hold at every point. Pick $x = \frac{\pi}{2n}$ and $y \neq 0$ so we have:

$$\left|u_1\left(\frac{\pi}{2n}, y\right) - u_2\left(\frac{\pi}{2n}, y\right)\right| = \frac{1}{n^2}|\sinh ny| = \frac{1}{2n^2}|e^{ny} - e^{-ny}|$$

Given that $y \neq 0$, one of the exponential terms will go to zero. Then we have that asymptotically as $n \to \infty$, we have:

$$\left|u_1\left(\frac{\pi}{2n}, y\right) - u_2\left(\frac{\pi}{2n}, y\right)\right| = \frac{1}{2n^2}e^{n|y|}$$

Note that this amount blows up as n increases. Hence the system is not well-posed. It is ill-posed.

Making a well-posed problem 1. Dirichlet boundary conditions $u = g$ on $\partial\Omega$, 2. Neumann Boundary Conditions, $\nabla u \cdot \vec{N} = \frac{\partial u}{\partial n} = g$ on $\partial\Omega$ or 3. Robin B.C.: $\alpha(x,y)u + \frac{\partial u}{\partial n} = g, \alpha > 0$ on $\partial\Omega$, 4. Mixed (combinations of the three above at different boundary positions). 5. For unbounded domains, we may require that the solution decays $u \to 0, |\vec{x}| \to \infty$.

3.3 Recitation 15 Apr 2015

Method of characteristics for fully nonlinear PDEs If the PDE is $F(Du, u, x) = 0$ then the characteristic equations are of the form:

$$\dot{\vec{p}}(s) = -D_{\vec{x}}F(\vec{p}, z, \vec{x}) - D_zF(\vec{p}, z, \vec{x})\vec{p}$$
$$\dot{z}(s) = D_{\vec{p}}F(\vec{p}, z, \vec{x}) \cdot \vec{p}$$
$$\dot{\vec{x}}(s) = D_{\vec{p}}F(\vec{p}, z, \vec{x})$$

Note that if the relation F was quasilinear or linear, then the $\vec{p}$ variable is not necessary and the first equation for $\dot{\vec{p}}$ vanishes.

Example $u_{x_1}u_{x_2} = u, x_1 > 0$ with initial data $u = x_2^2$ on $\Gamma = \{x_1 = 0\}$ (i.e. $u(0, x_2) = x_2^2$). Then we define the relation:

$$F(\vec{p}, z, \vec{x}) = p_1p_2 - z = 0$$

Then the method of characteristics gives:

$$\dot{p}_1 = p_1$$
$$\dot{p}_2 = p_2$$
$$\dot{z} = (p_2, p_1) \cdot (p_1, p_2) = 2p_1p_2 = 2z$$
$$\dot{x}_1 = p_2$$
$$\dot{x}_2 = p_1$$

Now we can solve for p_1, p_2, z:

$$\begin{aligned} p_1(s) &= p_1(0)e^s \\ p_2(s) &= p_2(0)e^s \\ z(s) &= z(0)e^{2s} \end{aligned}$$

and we pick (x_1^0, x_2^0) to lie on Γ, so that $x_1^0 = 0$. Then we have that:

$$z(0) = (x_2^0)^2$$

Now we want to relate $p_1(0)$ and $p_2(0)$. Note that we know the gradient of u along the line $x_1 = 0$, which is $u_{x_2}(0, x_2) = 2x_2$. Hence,

$$p_2(0) = 2x_2^0$$

Also, at the initial point $(0, x_2^0)$, the PDE requires that:

$$p_1(0)p_2(0) = (x_2^0)^2$$

and hence we have the expression for $p_1(0)$:

$$p_1(0) = \frac{x_2^0}{2}$$

Now we can solve for the characteristic coordinates:

$$\begin{aligned} \dot{x}_1 = p_2 = p_2(0)e^s = 2x_2^0 e^s, x_1^0 = 0 &\implies x_1(s) = 2x_2^0 e^s - 2x_2^0 \\ \dot{x}_2 = p_1 = p_1(0)e^s = \frac{x_2^0}{2}e^s, x_2(0) = x_2^0 &\implies x_2(s) = \frac{x_2^0}{2}e^s + \frac{x_2^0}{2} \end{aligned}$$

We now can eliminate x_2^0:

$$\begin{aligned} x_2^0 &= \frac{x_1}{2e^s - 2} \\ x_2^0 &= \frac{2x_2}{e^s + 1} \\ \implies \frac{x_1}{2e^s - 2} &= \frac{2x_2}{e^s + 1} \\ \implies x_1 e^s + x_1 &= 4x_2 e^s - 4x_2 \\ \implies e^s &= \frac{x_1 + 4x_2}{4x_2 - x_1} \end{aligned}$$

and we can substitute this into the expression for $z(s)$:

$$z(s) = z(0)e^{2s} = (x_2^0)^2 \left(\frac{x_1 + 4x_2}{4x_2 - x_1}\right)^2$$

Now we need to replace x_2^0. Note that:

$$x_2^0 = \frac{2x_2}{\frac{x_1 + 4x_2}{4x_2 - x_1} + 1} = \frac{2x_2(4x_2 - x_1)}{x_1 + 4x_2 + 4x_2 - x_1} = \frac{8x_2^2 - 2x_1x_2}{8x_2} = x_2 - \frac{x_1}{4}$$

Hence the solution is:

$$u(x, y) = (x_2 - \frac{x_1}{4})^2 \left(\frac{x_1 + 4x_2}{4x_2 - x_1}\right)^2 = \frac{1}{16}(x_1 + 4x_2)^2$$

3.4 Friday 17 Apr 2015

Laplace's Equation with Dirichlet Conditions Let $\Delta u = 0$ on Ω and $u = g$ on $\partial\Omega$. We examine the uniqueness condition by letting there be two solutions u_1, u_2 to the problem. Consider the difference: $u = u_1 - u_2$. To demonstrate uniqueness, we want to show that $u = 0$ on the closure of Ω which we denote $\bar{\Omega} = \Omega \cup \partial\Omega$. Now u solves Laplace's equation in Ω because the PDE is linear, but the boundary data this time is $u = 0$ on $\partial\Omega$. Now we multiply the PDE by u:

$$u\Delta u = 0$$

and we can integrate over all Ω:

$$\int_\Omega u\Delta u dV = 0$$

Now we apply Green's identity:

$$\int_\Omega u\Delta u dV = -\int_\Omega |\nabla u|^2 dV + \int_{\partial\Omega} u\frac{\partial u}{\partial N} d\sigma$$

where $\vec{N}$ is the outward unit normal at the boundary $\partial\Omega$. Now we know that $u = 0$ on the boundary. Hence we have:

$$-\int_\Omega |\nabla u|^2 dV = 0$$

But the square is non-negative. Hence we require that:

$$|\nabla u| = 0 \text{ on } \Omega$$

which implies that u is constant on the domain. But we know that u is zero on the boundary. Hence u is zero everywhere in the closure of the domain $\bar{\Omega}$. But this means that $u_1 = u_2$ on the domain, which implies that the solution must be unique.

Laplace's Equation with Neumann conditions Consider the PDE problem:

$$\Delta u = 0, \quad \frac{\partial u}{\partial N} = g$$

where the boundary data is defined on $\partial\Omega$. Note that if u is a solution, then $u + c$, where c is a constant is a solution, because the boundary is only concerned with the derivative of u at the boundary. Hence to ensure uniqueness, we need to impose an additional condition:

$$\int_\Omega u dV = m$$

which is a condition on the mean of u on the domain. We call this the mean constraint. Note that this constraint ensures that $u + c$ is no longer a solution if u is a solution, because $u + c$ will violate the mean constraint.

Poisson's Equation with Neumann conditions Consider the problem:

$$\Delta u = f, \quad \frac{\partial u}{\partial N} = g$$

Now f and g are not arbitrary. Note that:

$$\int_{\Omega} f dV = \int_{\Omega} \Delta u dV = \int_{\partial\Omega} \frac{\partial u}{\partial N} d\sigma = \int_{\partial\Omega} g d\sigma$$

where we use the divergence theorem (noting that $\Delta u = \nabla \cdot (\nabla u)$ and $\nabla u \cdot \vec{N} = \frac{\partial u}{\partial N}$). Hence the full Poisson equation and its conditions are:

$$\Delta u = f, \quad \frac{\partial u}{\partial N} = g \quad \int_{\Omega} f dV = \int_{\partial\Omega} g d\sigma$$

Note that if we consider the Laplace equation with Neumann conditions, then $f = 0$, and we have to satisfy:

$$0 = \int_{\partial\Omega} g d\sigma$$

Physical Example Let u be the temperature, ∇u be the heat flow, f be the source term, g be the value of u at the boundary. Then the rate of heat energy entering the domain is:

$$-\int_{\Omega} f dV$$

and the net flow along the boundary is:

$$\int_{\partial\Omega} g d\sigma$$

Since we have a steady state problem, we have the equality:

$$-\int_{\Omega} f dV + \int_{\partial\Omega} g d\sigma = 0$$

which is identical to the compatibility condition on Poisson's equation with Neumann boundary conditions.

Neumann Boundary Condition Uniqueness Let there be two solutions u_1, u_2. Then define $u = u_1 - u_2$. Then this function solves:

$$\Delta u = 0$$
$$\frac{\partial u}{\partial N} = 0$$
$$\int_{\Omega} u dV = 0$$

We repeat the uniqueness proof from above, multiplying the PDE by u and integrating over the domain to obtain:

$$\int_{\Omega} u \Delta u dV = -\int_{\Omega} |\nabla u|^2 dV + \int_{\partial\Omega} u \frac{\partial u}{\partial n} d\sigma = -\int_{\Omega} |\nabla u|^2 dV = 0$$

where the normal derivative vanishes along the boundary based on the boundary conditions. Hence we have that:

$$|\nabla u| = 0 \implies u = \text{constant}$$

and we know that $\int_{\Omega} u dV = 0$ and hence the mean of u is zero. But since u is constant on the closure of the domain, this means that u is zero everywhere in the domain.

Robin Boundary Condition and Uniqueness Now consider the problem:

$$\Delta u = 0 \text{ on } \Omega$$
$$\alpha(x,y)u + \frac{\partial u}{\partial \Omega} = g \text{ on } \partial\Omega, \alpha(x,y) > 0$$

Now assume that there are two solutions u_1, u_2 and take their difference $u = u_1 - u_2$. Then the difference satisfies:

$$\Delta u = 0 \text{ on } \Omega$$
$$\alpha u + \frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega$$

and we multiply the PDE by u, integrate over the whole domain, and apply Green's theorem to obtain:

$$\int_\Omega u\Delta u dV = -\int_\Omega |\nabla u|^2 dV + \int_{\partial\Omega} u\frac{\partial u}{\partial n} d\sigma = 0$$

Substituting the Robin BC,

$$\int_\Omega u\Delta u dV = -\int_\Omega |\nabla u|^2 dV + \int_{\partial\Omega} u(-\alpha u) d\sigma = 0$$

But we know that $\alpha(x,y) > 0$ on the domain, which means that we are integrating two terms that are strictly negative. The equality to zero means that:

$$\int_\Omega |\nabla u|^2 dV = -\int_{\partial\Omega} \alpha u^2 d\sigma$$

But this is a contradiction unless:

$$|\nabla u| = 0 \text{ on } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

and we have that $u = 0$ on Ω immediately. Hence we have uniqueness for this boundary condition.

Energy of system Define the energy of the system:

$$E(u) \equiv \int_\Omega |\nabla u|^2 dV$$

2D forms of Laplace's equation Consider the 2D problem in Cartesian and Polar coordinates

$$u_{xx} + u_{yy} = 0$$
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

n-D Polar coordinates Define:

$$\Delta u = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\Lambda_{n-1}u = 0$$

where Λ represents an operator involving just angular derivatives. We will just concern ourselves with the radial part. Make the assumption that the solution is radially symmetric. $u = u(r)$.

Radially symmetric Laplace Equation Consider:

$$\Delta u = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) = 0$$

on the domain $r > 0$. Where the partial derivatives are actually single variable derivatives. Hence we can eliminate the $\frac{1}{r^{n-1}}$ part and obtain that:

$$r^{n-1}\frac{\partial u}{\partial r} = c, c \in \mathbb{C}$$

In 2D, this has the form:

$$\frac{\partial u}{\partial r} = \frac{c}{r}$$

where we treat $2D$ separately because the integral in 2D will be logarithmic. Then the solution has the form:

$$u(r) = \begin{cases} c_1 \ln(r) + c_2, & n = 2 \\ \frac{c_1}{r^{n-2}} + c_2, n \neq 2 \end{cases}$$

Extended domain to include origin The original domain of the problem is $\mathbb{R}^n \backslash \{0\}$. We can extend the domain to $\mathbb{R}^n$ including the origin by introducing:

$$\Delta u(r) = c\delta(r)$$

The delta function satisfies:

$$\int_{\mathbb{R}^n} \delta(z)dz = 1$$
$$\int_{\mathbb{R}^n} f(x)\delta(x - x_0)dV = f(x_0)$$

Consider an n-ball centered at zero with radius ϵ. Call it $B(0, \epsilon)$. Then we integrate the Laplacian in the ball and apply the divergence theorem:

$$\int_B \Delta u dV = \int_{\partial B} \frac{\partial u}{\partial N} d\sigma$$

Now the normal vector of a ball at the origin is just the radial unit vector. Hence the normal derivative is just the radial derivative:

$$\int_B \Delta u dV = \int_{\partial B} \frac{\partial u}{\partial N} d\sigma = \int_{\partial B} \frac{\partial u}{\partial r} d\sigma$$

But we know that the radial derivative is just:

$$\frac{\partial u}{\partial r} = \frac{c}{r^{n-1}}$$

by examining the form of the Laplacian in n-D polar coordinates. Then we have:

$$\int_B \Delta u dV = \int_{\partial B} \frac{c}{r^{n-1}} d\sigma$$

But we know that the surface element $d\sigma$ must scale as r^{n-1}. Hence we may cancel r^{n-1} to obtain that the RHS must be a constant. Hence we have:

$$\int_B \Delta u dV = \text{constant}$$

and this is valid for all n-balls for all radii ϵ. Hence the integral is a constant, which means that there must be a delta function at the origin. This motivates us to define the new function (refer to Week 4 notes for the correct sign convention):

$$F(r) = \begin{cases} \frac{1}{2\pi} \ln r, & n = 2 \\ \frac{-1}{4\pi r}, & n = 3 \end{cases}$$

so that the integral of $F(r)$ for a small n-ball at the origin is normalized and gives unity. Note that $\Delta F(r) = \delta(r)$

Chapter 4

Week 4

4.1 Monday 20 Apr 2015

Radial Solutions to Laplace Equation Recall that:

$$F(r) = \begin{cases} -\frac{1}{2\pi}\ln(|r|), & n = 2 \\ \frac{1}{4\pi}\frac{1}{|r|}, & n = 3 \end{cases}$$

and F is the fundamental solution to Laplace's equation (note the minus sign):

$$-\Delta F(r) = \delta(r)$$

Now define:

$$F(\vec{p}) = F(|\vec{p}|), \vec{p} \in \mathbb{R}^n$$

Now we want to solve the problem on the bounded connected domain Ω:

$$\begin{aligned} \Delta u = f, &\quad \text{in } \Omega \subset \mathbb{R}^n \\ u = g &\quad \text{on } \partial\Omega \end{aligned}$$

We proceed by an intuitive method. We know that at any point $\vec{p} \in \mathbb{R}^n$, the negative Laplacian of the function F will be the Dirac delta. Now we can shift the coordinate system by the constant vector $\vec{p}_0$:

$$-\Delta F(\vec{p} - \vec{p}_0) = \delta(\vec{p} - \vec{p}_0)$$

Now apply Green's second identity:

$$\int_\Omega (u\Delta w - w\Delta u) dV = \int_{\partial\Omega} \left(u\frac{\partial w}{\partial N} - w\frac{\partial u}{\partial N} \right) d\sigma$$

We take $u \to F(\vec{p} - \vec{p}_0)$. Then:

$$\int_\Omega \left[u(\vec{p})\Delta F(\vec{p} - \vec{p}_0) - F(\vec{p} - \vec{p}_0)\Delta u(\vec{p})\right] dV = \int_{\partial\Omega} \left(u(\vec{p})\frac{\partial F(\vec{p} - \vec{p}_0)}{\partial N} - F(\vec{p} - \vec{p}_0)\frac{\partial u(\vec{p})}{\partial N} \right) d\sigma$$

The trick is that we know the second derivative of F. Hence we replace $-\Delta F = \delta$ and $\Delta u = f$:

$$\int_\Omega [-u(\vec{p})\delta(\vec{p}-\vec{p}_0) - F(\vec{p}-\vec{p}_0)f(\vec{p})]\, dV = \int_{\partial\Omega} \left(u(\vec{p})\frac{\partial F(\vec{p}-\vec{p}_0)}{\partial N} - F(\vec{p}-\vec{p}_0)\frac{\partial u(\vec{p})}{\partial N} \right) d\sigma$$

The first term of the LHS is the integral of a delta function:

$$\begin{aligned}
-u(\vec{p}_0) &= \int_\Omega [F(\vec{p}-\vec{p}_0)f(\vec{p})]\, dV + \int_{\partial\Omega} \left(u(\vec{p})\frac{\partial F(\vec{p}-\vec{p}_0)}{\partial N} - F(\vec{p}-\vec{p}_0)\frac{\partial u(\vec{p})}{\partial N} \right) d\sigma \\
\implies u(\vec{p}_0) &= -\int_\Omega [F(\vec{p}-\vec{p}_0)f(\vec{p})]\, dV + \int_{\partial\Omega} \left(F(\vec{p}-\vec{p}_0)\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial F(\vec{p}-\vec{p}_0)}{\partial N} \right) d\sigma
\end{aligned}$$

This result is known as the **Representation Theorem**. Once we know the Laplacian of the function (i.e. f), we just need the values of the function at the boundary (both Neumann and Dirichlet).

2D/3D Representation Theorem

$$u(\vec{p}_0) = \frac{1}{2\pi}\int \ln|\vec{p}-\vec{p}_0| f(\vec{p}) dV - \frac{1}{2\pi}\int_{\partial\Omega} \left(\ln|\vec{p}-\vec{p}_0|\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial F(\ln|\vec{p}-\vec{p}_0|)}{\partial N} \right) d\sigma$$

and similarly in 3D:

$$u(\vec{p}_0) = -\frac{1}{4\pi}\int \frac{f(\vec{p})}{|\vec{p}-\vec{p}_0|} dV + \frac{1}{4\pi}\int \left(\frac{1}{|\vec{p}-\vec{p}_0|}\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial}{\partial N}\frac{1}{|\vec{p}-\vec{p}_0|} \right) d\sigma$$

Unbounded domain boundary condition If $\Omega = \mathbb{R}^n$, the whole space, we implement the boundary condition that:

$$u \to 0, |\nabla u| \to 0 \quad \text{as } |\vec{p}| \to \infty$$

Then the terms on the boundary of the representation theorem vanish, and we can write:

$$u(\vec{p}_0) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|\vec{p}-\vec{p}_0| f(\vec{p}) dV, & n=2 \\ -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\vec{p})}{|\vec{p}-\vec{p}_0|} dV, & n=3 \end{cases}$$

Note that the fundamental solution F is the Green's function if the domain is unbounded, but it is not the Green's function if the domain is finite.

Mean Value Property Consider Laplace's equation with Dirichlet Boundary data on a bounded, connected domain Ω with smooth boundary $\partial\Omega$. Now pick a point $\vec{p}_0 \in \Omega$ and draw and n-ball of radius ϵ such that the ball is contained in the domain. Call this ball $B(\vec{p}_0, \epsilon)$. Now the function must be harmonic on the ball as well, $\Delta u = 0$. We apply the representation theorem directly to the n-ball. Note that when the function is Harmonic, the first term in the theorem drops out because $f = 0$, so we just need to integrate on the boundary. WLOG assume $n = 3$. Then:

$$u(\vec{p}_0) = \frac{1}{4\pi}\int_{S(\vec{p}_0,\epsilon)} \left(\frac{1}{|\vec{p}-\vec{p}_0|}\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial}{\partial N}\frac{1}{|\vec{p}-\vec{p}_0|} \right) d\sigma$$

Now since $\vec{p} \in S(\vec{p}_0, \epsilon)$, we know that

$$|\vec{p}-\vec{p}_0| = \epsilon$$

since it is a sphere. Also, the partial derivative:

$$\frac{\partial}{\partial N}\frac{1}{|\vec{p}-\vec{p}_0|} = -\frac{1}{\epsilon^2}$$

when evaluated at the boundary of the sphere. Hence we write:

$$u(\vec{p}_0) = \frac{1}{4\pi}\int_{S(\vec{p}_0,\epsilon)} \left(\frac{1}{\epsilon}\frac{\partial u(\vec{p})}{\partial N} + u(\vec{p})\frac{1}{\epsilon^2}\right) d\sigma$$

Now the first term involves the normal derivative of the function integrated along the boundary. But the function is harmonic, hence by the divergence theorem, the first integral involves the Laplacian and it vanishes. Hence we write:

$$u(\vec{p}_0) = \frac{1}{4\pi\epsilon^2}\int_{S(\vec{p}_0,\epsilon)} u(\vec{p})d\sigma$$

Note that $4\pi\epsilon^2$ is the surface area of the 3-sphere. Hence the value of the function at the point $\vec{p}_0$ is just the average of the values along the boundary:

$$u(\vec{p}_0) = \frac{1}{|S(\vec{p}_0,\epsilon)|}\int_{S(\vec{p}_0,\epsilon)} u(\vec{p})d\sigma$$

the above equation holds for any dimension.

Maximum principle Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose that u is continuous in $\bar{\Omega} = \Omega \cup \partial\Omega$ and harmonic in Ω. Then u takes its maximum and minimum value on $\partial\Omega$. If u is not constant, then u attains its maximum and minimum only on $\partial\Omega$.

Intuition behind Maximum Principle Proceed by contradiction. Assume that u is not constant and has a maximum value M on the closure of the domain, and M does not occur on the boundary. Then there exists a point $\vec{p}_0 \in \Omega\backslash\partial\Omega$ such that $u(\vec{p}_0) = M$. Then pick a ball centered at $\vec{p}$ with radius R_0 such that it just touches the boundary. Then we have, from the mean value property,

$$u(\vec{p}_0) = \frac{1}{|\partial B(\vec{p}_0, R_0)|}u(\vec{p})d\sigma$$

But we have that $u(\vec{p}_0) = M \geq u(\vec{p})$. Since M was the average of the values along the boundary, and M is also the maximum of the values along the boundary, the function must take M along the boundary as well. Since this argument applies for any $R \in [0, R_0]$, we have that $u(\vec{p}) = M$ for $\vec{p} \in B(\vec{p}_0, R)$. Now pick another position in the ball, make the same argument and repeat until you cover the whole domain. Then the function must be constant. Contradiction. Hence M cannot occur in the interior of the domain.

Consequences of the Maximum Principle for Laplace Equation

- If $u \geq 0$ on $\partial\Omega$ then $u \geq 0$ on Ω.
- If $u \leq 0$ on $\partial\Omega$ then $u \leq 0$ on Ω.
- Uf $u = c$ on $\partial\Omega$ then $u = c$ on Ω.
- Continuous dependence on initial data: $|u(\vec{p})|$ is bounded above by $\max_{\partial\Omega}|g|$ and bounded below by $\min_{\partial\Omega}|g|$. This can be extended to the Poisson problem.

Poisson Equation Bound Consider $\Delta u = f, u = g$. Then the bound for the solution is:

$$|u(\vec{p})| \leq \max_{\partial\Omega}|g| + C\max_{\Omega}|f|$$

where C is a constant depending on the size of the domain. The larger the domain, the larger C is.

Liouville's Theorem A non-constant harmonic function in an entire domain (any dimension) cannot be bounded from above or below.

Corollary of Liouville's theorem The only bounded harmonic functions on the entire domain are constant functions.

4.2 22 Apr 2015 Wednesday

Review of Representation Theorem

$$u(\vec{p}_0) = -\int_{\Omega} [F(\vec{p}-\vec{p}_0)f(\vec{p})]\, dV + \int_{\partial\Omega} \left(F(\vec{p}-\vec{p}_0)\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial F(\vec{p}-\vec{p}_0)}{\partial N} \right) d\sigma$$

Using Representation Theorem to solve Dirichlet and Neumann B.C. To use the Representation Theorem as a solution, we require these pieces of infomation:

$$\Delta u = f, \quad u \text{ on } \partial\Omega, \quad \frac{\partial u}{\partial N} \text{ on } \partial\Omega$$

It will be good if we could eliminate one of these pieces of information using the third.

First consider the Dirichlet B.C. Then we knew $\Delta u = f$ and $u = g$ on $\partial\Omega$. Also, the fundamental solution F was defined on $r \geq 0$ and $\theta \in [0, 2\pi]$. Now we want to obtain the Green's function G for the problem:

$$-\Delta_{\vec{p}} G(\vec{p}, \vec{p}_0) = \delta(\vec{p}-\vec{p}_0)$$

under the constraint that:

$$G(\vec{p}, \vec{p}_0) = 0, \quad \forall \vec{p} \in \partial\Omega$$

Substituting this Green's function into the representation theorem (replacing F),

$$u(\vec{p}_0) = -\int_{\Omega} G(\vec{p}, \vec{p}_0)f(\vec{p})dV - \int_{\partial\Omega} u(\vec{p})\frac{\partial}{\partial N}G(\vec{p}, \vec{p}_0)d\sigma = -\int_{\Omega} G(\vec{p}, \vec{p}_0)f(\vec{p})dV - \int_{\partial\Omega} g(\vec{p})\frac{\partial}{\partial N}G(\vec{p}, \vec{p}_0)d\sigma$$

because $u(\vec{p}) = g(\vec{p})$ on $\partial\Omega$. Note that the Neumann condition is no longer relevant because it was multiplied into G, which was chosen to vanish on the boundary. Also, if u is purely harmonic, then $f = 0$ and we just have:

$$u(\vec{p}_0) = -\int_{\partial\Omega} g(\vec{p})\frac{\partial}{\partial N}G(\vec{p}, \vec{p}_0)d\sigma$$

Hence once we have the Green's function, we can construct the solution of the Equation with just two pieces of information (the value of the source term and the value of the function along the boundary).

Constructing the Green's function Consider writing the Green's function as the sum of two functions:

$$G = F + H$$

where F is the fundamental solution in $\mathbb{R}^n$. But since both G and F have the same Laplacian (delta function), H must be purely harmonic:

$$\Delta H(\vec{p}, \vec{p}_0) = 0 \text{ in } \Omega$$
$$H(\vec{p}, \vec{p}_0) = -F(\vec{p}-\vec{p}_0) \text{ on } \partial\Omega$$

because

$$-\Delta G = \delta(\vec{p}-\vec{p}_0) = -\Delta F - \Delta H = \delta(\vec{p}-\vec{p}_0) - \Delta H \implies \Delta H = 0$$

and on the boundary $\partial\Omega$:

$$G = F + H = 0 \implies H = -F$$

Recall we already had an expression for F on $\mathbb{R}^n$:

$$F(\vec{p} - \vec{p}_0) = \begin{cases} -\frac{1}{2\pi}\ln(|\vec{p} - \vec{p}_0|), & n = 2 \\ \frac{1}{4\pi}\frac{1}{|\vec{p} - \vec{p}_0|}, & n = 3 \end{cases}$$

Note that F is harmonic in Ω as long as $\vec{p}_0 \notin \bar{\Omega}$, because its Laplacian is the delta function, and away from $\vec{p}_0$, it has value zero.

Construction of G in 2D Consider the problem $\Omega = \{(x, y)|y > 0\} \subset \mathbb{R}^2$, which is the upper half plane. Now we want to solve Laplace's equation with Dirichlet boundary conditions:

$$\begin{aligned} \Delta u(x, y) &= 0, \quad y > 0 \\ u(x, 0) &= g(x), \quad y = 0 \end{aligned}$$

Then the Green's function satisfies:

$$\begin{aligned} G(\vec{p}, \vec{p}_0) &= -\frac{1}{2\pi}\ln|\vec{p} - \vec{p}_0| + H(\vec{p}, \vec{p}_0) \\ \Delta H &= 0 \quad \text{in } \Omega \\ H(\vec{p}, \vec{p}_0) &= -F = \frac{1}{2\pi}\ln|\vec{p} - \vec{p}_0|, \quad \vec{p} \in \partial\Omega \end{aligned}$$

Analogy to Electrostatics Consider the upper half space in any dimension. Consider a point $\vec{p}_0$ in the upper half space, and place a charged particle there. Then we can use the method of images to find the potential in all space. The potential emitted by the first particle is given by $F(\vec{p}, \vec{p}_0)$ and the potential emitted by the second particle will be $-F(\vec{p}, \vec{p}_0)$ such that the net effect on the potential on the boundary separating the half-spaces, which is $\partial\Omega$, will be zero, just like how $G(\vec{p}, \vec{p}_0)$ vanishes at the boundary.

Going back to the 2D Dirichlet problem We write the potential H as

$$H(\vec{p}, \vec{p}_0) = \frac{1}{2\pi}\ln|\vec{p} - h(\vec{p}_0)|, \quad \vec{p}_0^* \notin \Omega$$

where $h(\vec{p}_0) \equiv \vec{p}_0^*$ is some function that maps $\vec{p}_0$ out of the domain so we know that this function is harmonic on Ω because the point $h(\vec{p}_0)$ is not contained in Ω. We implement the boundary condition:

$$\frac{1}{2\pi}\ln|\vec{p} - \vec{p}_0^*| = \frac{1}{2\pi}\ln|\vec{p} - \vec{p}_0|, \quad \vec{p} \in \partial\Omega$$

But this means that:

$$\begin{aligned} |\vec{p} - \vec{p}_0^*| &= |\vec{p} - \vec{p}_0|, \quad \vec{p} \in \partial\Omega \\ \text{Squaring and expanding the dot product,} \implies |\vec{p}|^2 - 2\vec{p}\cdot\vec{p}_0^* + |\vec{p}_0^*|^2 &= |\vec{p}|^2 - 2\vec{p}\cdot\vec{p}_0 + |\vec{p}_0|^2 \\ \implies |\vec{p}_0|^2 - |\vec{p}_0^*|^2 &= 2\vec{p}\cdot(\vec{p}_0 - \vec{p}_0^*) \end{aligned}$$

But on the boundary, $y = 0$, hence we have that:

$$\vec{p} = (p, 0)$$

Now we have:

$$|\vec{p_0}|^2 - |\vec{p_0^*}|^2 = 2(p, 0) \cdot (\vec{p_0} - \vec{p_0^*})$$

Now the LHS does not depend on p, a variable, but the RHS does. To ensure that equality holds for all $\vec{p}$, we hence have that (taking $p = 1$ for instance just to remove the trivial zero equality):

$$\vec{p} \cdot (\vec{p_0} - \vec{p_0^*}) = (p, 0) \cdot ((x_0, y_0) - (x_0^*, y_0^*)) = 0 \quad \vec{p} \in \partial\Omega \implies x_0 = x_0^* \implies \vec{p_0^*} = (x_0, y_0^*)$$

Then examining the LHS,

$$|\vec{p_0}|^2 = |\vec{p_0^*}|^2 \implies x_0^2 + y_0^2 = x_0^2 + (y_0^*)^2 \implies y_0^* = \pm y_0$$

Now if $y_0^* = y_0$, the function will not be harmonic. y_0^* cannot be contained inside the domain, since the Laplacian is a delta function centred at y_0^*. Hence we have:

$$y_0^* = -y_0$$
$$H(\vec{p}, \vec{p_0}) = \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y + y_0)^2}$$

and we can construct Green's function:

$$G(\vec{p}, \vec{p_0}) = G((x, y), (x_0, y_0)) = -\frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} + \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y + y_0)^2}$$

Example 2 for 3D space Let $\Omega = \{(x, y, z) | z > 0\} \subset \mathbb{R}^3$ and $u(x, y, 0) = g(x, y)$ for the boundary $\partial\Omega = \{(x, y, z) | z = 0\}$. Then the Green's function is the fundamental solution plus the harmonic part H:

$$G(\vec{p}, \vec{p_0}) = F(\vec{p}, \vec{p_0}) + H(\vec{p}, \vec{p_0}) = \frac{1}{4\pi} \frac{1}{|\vec{p} - \vec{p_0}|} + H(\vec{p}, \vec{p_0^*})$$

Going through the same proof as above, we can obtain that the image charge will be at:

$$\vec{p_0^*} = (x_0, y_0, -z_0)$$

where we have defined $\vec{p_0} = (x_0, y_0, z_0)$. Hence:

$$H(\vec{p}, \vec{p_0}) = -F(\vec{p}, -\vec{p_0^*})$$

Explicitly,

$$G(\vec{p}, \vec{p_0}) = \frac{1}{4\pi} \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} - \frac{1}{4\pi} \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}}$$

Since we have the Green's function and the source term was zero, we can now construct the solution to the Dirichlet problem:

$$u(\vec{p}_0) = u(x_0, y_0, z_0) = -\int_{z=0} g(x,y)\frac{\partial}{\partial N}G((x,y,z),(x_0,y_0,z_0))d\sigma$$

Now the normal vector in the 3D half space is just the negative z unit vector:

$$\vec{N} = (0,0,-1) \implies \frac{\partial}{\partial N} = -\frac{\partial}{\partial z}$$

and we can calculate the normal derivative of the Green's function:

$$\left.\frac{\partial G}{\partial N}\right|_{\partial\Omega} = -\left.\frac{\partial G}{\partial z}\right|_{\partial\Omega} = \left.\left(\frac{-1}{4\pi}\frac{z-z_0}{|\vec{p}-\vec{p}_0|^3} + \frac{1}{4\pi}\frac{z+z_0}{|\vec{p}-\vec{p}_0^*|^3}\right)\right|_{\partial\Omega}$$

Along the boundary, all the $z = 0$. Hence we have:

$$\left.\frac{\partial G}{\partial N}\right|_{\partial\Omega} = -\frac{1}{4\pi}\frac{-z_0}{|(x,y,0)-(x_0,y_0,z_0)|^3} + \frac{1}{4\pi}\frac{z_0}{|(x,y,0)-(x_0,y_0,-z_0)|^3} = \frac{1}{2\pi}\frac{z_0}{((x-x_0)^2+(y-y_0)^2+z_0^2)^{3/2}}$$

Substituting this expression into the Representation theorem form of the solution:

$$u(x_0,y_0,z_0) = \frac{-z_0}{2\pi}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\frac{g(x,g)dxdy}{((x-x_0)^2+(y-y_0)^2+z_0^2)^{3/2}}$$

Using maximum principle to show uniqueness Consider the problem:

$$\Delta u = f \text{ on } \Omega$$
$$u = g \text{ on } \partial\Omega$$

Define $u = u_1 - u_2$, where u_1 and u_2 are two solutions to the equation. Then u satisfies:

$$\Delta u = 0 \text{ on } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Now by the Maximum Principle,

$$\min_{\partial\Omega} u \le u \le \max_{\partial\Omega} u$$
$$\implies 0 \le u \le 0 \implies u = 0$$

and the solution must be unique. QED.

4.3 Friday 24 Apr 2015

Separation of Variables Consider the problem: $\Omega = [0,a]\times[0,b]$. We want to find the function that is harmonic $u_{xx}+u_{yy} = 0$ and satisfies Dirichlet boundary conditions:

$$u(x,0) = g_1(x)$$
$$u(x,b) = g_2(x)$$
$$u(0,y) = g_3(y)$$
$$u(a,y) = g_4(y)$$

By linearity, we can write u as the sum of four functions:

$$u = u_1 + u_2 + u_3 + u_4$$
$$\Delta u_j = 0, j = 1, 2, 3, 4$$

that satisfies the boundary conditions:

$$\begin{aligned} u_1(x, 0) &= g_1(x), \quad \text{0 at other boundaries} \\ u_2(x, b) &= g_2(x), \quad \text{0 at other boundaries} \\ u_3(0, y) &= g_3(x), \quad \text{0 at other boundaries} \\ u_4(a, y) &= g_4(x), \quad \text{0 at other boundaries} \end{aligned}$$

Effectively, each of the functions satisfies one of the boundary conditions and is zero on the other boundaries. Hence the sum of these functions will satisfy the boundary condition at each of the boundaries.

Separation of Variables Assumption For $\Delta U = 0$ in $\Omega = [0, a] \times [0, b]$, write $U(x, y) = X(x)Y(y)$. Then:

$$U_{xx} = X''Y, \quad U_{yy} = XY'' \implies X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\mu$$
$$\implies X'' - \mu X = 0, \quad Y'' + \mu Y = 0$$

If $\mu = 0$, then $X \sim \{1, x\}$ and $Y \sim \{1, y\}$ where this notation indicates that X and Y are linear combinations of a constant and x or y.

If $\mu \neq 0$, then write $\mu = -\lambda^2$, where $\lambda \in \mathbb{C}$. Then the solutions are:

$$X(x) \sim \{\sin(\lambda x), \cos(\lambda x)\}$$
$$Y(y) \sim \{\sinh(\lambda y), \cosh(\lambda y)\}$$

Note that we also can write $\mu = \lambda^2$ which will give solutions with the trig and hyperbolic parts switched.

Then the original function U can be written as a linear combination as well:

$$U \sim \{1, x, y, xy\}, \quad \mu = 0$$
$$U \sim \{\cos(\lambda x)\cosh(\lambda y), \cos(\lambda x)\sinh(\lambda y), \sin(\lambda x)\cosh(\lambda y), \sin(\lambda x)\sinh(\lambda y)\}, \quad \mu = -\lambda^2$$
$$U \sim \{\cosh(\lambda x)\cos(\lambda y), \cosh(\lambda x)\sin(\lambda y), \sinh(\lambda x)\cos(\lambda y), \sinh(\lambda x)\sin(\lambda y)\}, \quad \mu = \lambda^2$$

Now consider the problem:

$$\Delta u_1 = 0, \quad u_1(x, 0) = g_1(x)$$

and zero along the other boundaries. Now the hyperbolic functions can have at most one zero. Since we require that the function vanishes twice in the x-direction, we require that the solution be trigonometric (not hyperbolic) in x. Then the function must be hyperbolic in y. Then we write the conditions and a guess for the solution in X:

$$X(0) = X(a) = 0$$
$$X(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$$
$$\implies c_1 = 0, \quad c_1\cos(\lambda a) + c_2\sin(\lambda a) = 0 \implies \lambda = \frac{\pi n}{a}, n \in \mathbb{Z}$$

We disallow $c_2 = 0$ so that we do not have the trivial case. Then we have a family of solutions for X:

$$X_n(x) = \sin \lambda_n x, \quad \lambda_n = \frac{\pi n}{a}$$

Now for the y direction, we can implement a boundary condition at $y = b$, where the function must vanish:

$$Y_n(b) = 0 = c_3 \cosh(\lambda_n b) + c_4 \sinh(\lambda_n b)$$

for the same eigenvalues as before. Rearranging,

$$c_3 = -c_4 \tanh(\lambda_n b)$$

Then we can simplify the hyperbolic expression for y and obtain (up to a constant)

$$Y_n(y) = \sinh(\lambda_n(y - b))$$

Now we have a family of solutions for U:

$$U_n(x, y) = X_n(x)Y_n(y) = C \sin(\lambda_n x) \sinh(\lambda_n(y - b))$$

The general solution is a linear combination of these eigenfunctions:

$$u_1(x, y) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(y - b)$$

To pick α_n, we need to satisfy the boundary condition on the boundary:

$$u_1(x, 0) = g_1(x) \implies \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(0 - b) = g_1(x)$$

Defining $\hat{\alpha}_n = -\alpha_n \sinh \frac{n\pi b}{a}$, we have the sine series:

$$g_1(x) = \sum_{n=1}^{\infty} \hat{\alpha}_n \sin \frac{n\pi x}{a}$$

and hence:

$$\hat{\alpha}_n = \frac{2}{a} \int_0^a g_1(x) \sin \frac{n\pi x}{a} dx \implies \alpha_n = \frac{2}{a} \frac{-1}{\sinh \frac{n\pi b}{a}} \int_0^a g_1(x) \sin \frac{n\pi x}{a} dx$$

by Fourier's theorem. Then the solution is:

$$u_1(x, y) = \sum_{n=1}^{\infty} \frac{-1}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(y - b) \int_0^a g_1(x) \sin \frac{n\pi x}{a} dx$$

In general, if there are two zeros in one direction, we know that we have to use the trigonometric form. The other component functions are:

$$u_2(x,y) = \sum_{n=1}^{\infty} \beta_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}, \quad \beta_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a g_2(x) \sin \frac{n\pi x}{a} dx$$

$$u_3(x,y) = \sum_{n=1}^{\infty} \eta_n \sinh \frac{n\pi}{b}(x-a) \sin \frac{n\pi y}{b}, \quad \eta_n = \frac{-2}{b \sinh \frac{\pi n a}{b}} \int_0^b g_3(y) \sin \frac{n\pi y}{b} dy$$

$$u_4(x,y) = \sum_{n=1}^{\infty} \gamma_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \quad \gamma_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_4(y) \sin \frac{n\pi y}{b} dy$$

Chapter 5

Week 5

5.1 Monday 27 Apr 2015

Separation of Variables in Polar Coordinates Consider the problem in a bounded domain:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad \text{on } B(0,1) \subset \mathbb{R}^2$$
$$u(1,\theta) = g(\theta), \quad \text{on } S(0,1) = \partial B(0,1)$$

where we let $\theta \in [-\pi, \pi]$ and $r \in [0,1)$ for $B(0,1)$. Separating variables, $u(r,\theta) = R(r)\Theta(\theta)$ and

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

and hence each side must equal to a constant. Hence let the separation constant $\mu \in \mathbb{R}$, and:

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = \mu$$
$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu$$

which are ODEs. The angular equation is trivial:

$$\Theta(\theta) \sim \begin{cases} \{1, \theta\} & \mu = 0 \\ \{\cos(\sqrt{\mu}\theta), \sin(\sqrt{\mu}\theta), & \mu \neq 0 \end{cases}$$

But now we have to impose a periodicity condition. If we increase θ by 2π, we need the function to return to the same value:

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$

and this is a condition for the function to be harmonic, since the harmonic function must be continuous. Then we have:

$$\Theta(\theta) \sim \begin{cases} \{1\}, & \mu = 0 \\ \{\cos(n\theta), \sin(n\theta), n \in \mathbb{Z}^+, & \mu \neq 0 \end{cases}$$

so $\sqrt{\mu} = n$, and the functions are strictly trigonometric ($\sqrt{\mu}$ cannot be imaginary).
The radial equation is Euler's ODE,

$$r^2R'' + rR' - \mu R = 0$$

which we can solve:

$$R(r) \sim \begin{cases} \{1, \ln r\}, & \mu = 0 \iff n = 0 \\ \{r^n, r^{-n}\}, & \mu \neq 0 \iff n \geq 1 \end{cases}$$

Hence under the separation of variables assumption, we have solutions that look like:

$$u(r,\theta) = \frac{a_0}{2} + c_0\frac{\ln r}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] + \sum_{n=1}^{\infty} r^{-n} \left[c_n \cos(n\theta) + d_n \sin(n\theta)\right]$$

where the first term represents the $n = 0$.

About zero in the domain Since the origin is in the domain $B(0,1)$, we cannot allow the $\ln r$ terms and r^{-n} terms to remain in the solution, since the function will not be continuous and hence will not be harmonic if they are in the solution. Then the general form of the harmonic solution is:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

Imposing boundary data

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] = g(\theta)$$

and hence we can find the coefficients:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)d\theta, \quad n \geq 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)d\theta, \quad n \geq 1 \end{aligned}$$

This solution holds as long as g is continuous on $\theta \in [-\pi, \pi]$.

Donut-shaped domain/annulus If the origin is not in the domain (but the domain must still be bounded!), such as in the annulus domain, we can keep all the terms in the general solution:

$$u(r,\theta) = \frac{a_0}{2} + c_0\frac{\ln r}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] + \sum_{n=1}^{\infty} r^{-n} \left[c_n \cos(n\theta) + d_n \sin(n\theta)\right]$$

General disk Consider $\Omega = B(0,\rho), \rho > 0$. Then the problem is:

$$\begin{aligned} \Delta u(r,\theta) &= 0, \quad \text{in } B(0,\rho) \\ u(\rho,\theta) &= g(\theta) \end{aligned}$$

Consider the related problem:

$$\Delta v(r,\theta) = 0, \quad \text{in } B(0,1)$$
$$v(1,\theta) = g(\theta)$$

and we already know the solution for v:

$$v(r,\theta) = \frac{a_0}{2} + c_0 \frac{\ln r}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] + \sum_{n=1}^{\infty} r^{-n} \left[c_n \cos(n\theta) + d_n \sin(n\theta)\right]$$

Hence to obtain u from v, we just re-scale v in the r-direction:

$$u(r,\theta) = v(r/\rho,\theta) = \frac{a_0}{2} + c_0 \frac{\ln(r/\rho)}{2} + \sum_{n=1}^{\infty} (r/\rho)^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] + \sum_{n=1}^{\infty} (r/\rho)^{-n} \left[c_n \cos(n\theta) + d_n \sin(n\theta)\right]$$

which can be shown to be the solution by applying the chain rule or by making the change of variables $r' = r/\rho$.

Superharmonic and Subharmonic Define:

$$u \in C^2(\Omega) \text{ is } \begin{cases} \text{Harmonic if } -\Delta u = 0 \\ \text{Superharmonic if } -\Delta u \geq 0 \\ \text{Subharmonic if } -\Delta u \leq 0 \end{cases}$$

and if a function is super and subharmonic, then it is harmonic.

Maximum principle for super and subharmonic functions If u is subharmonic, then $u \leq \max_{\partial\Omega} u$. If u is superharmonic, then $u \geq \min_{\partial\Omega} u$. Hence for harmonic functions, both conditions hold.

Poisson's Equation Consider:

$$-\Delta u_j = f_j, \quad j = 1,2$$
$$u_j = g$$

so we are looking for two solutions u_1, u_2 with different source terms but subject to the same boundary conditions. Take the difference of the solutions. Assume $f_1 \geq f_2$ on the domain. Then:

$$-\Delta(u_1 - u_2) = f_1 - f_2 \geq 0$$

and $u_1 - u_2$ is superharmonic.

Application: Error Bounds Suppose there is a force f that is very noisy. Then we can approximate it by defining a smooth force that bounds the force from below and another smooth force that bounds the force from above. Then solving the problem with each of the smooth forces, we know that the true solution is bounded between each of the smooth approximations.

Relationship of Laplace's Equation to Linear Algebra Let $\boldsymbol{M}$ be an $n \times n$ matrix. Given a point $\vec{x} \in \Omega$, the Hessian is a matrix:

$$Hu(\vec{x}) = \left(u_{x_i,x_j}\right)_{i,j=1}^{n}$$

hence the trace of the Hessian is the Laplacian. Hence we can rotate and distort the coordinate system and the trace will be invariant. Also, the trace is the sum of the eigenvalues of the Hessian at a point.

5.2 29 Apr 2015

Parity: Even or Odd Recall the Fourier Series:

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin(nx)dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos(nx)dx$$

If $u(x)$ is odd, then $a_n = 0$. If $u(x)$ is even, then $b_n = 0$.

Gibbs Phenomena When the original function is discontinuous, there will be an overshoot for the Fourier series. But when N goes to infinity, the over/undershoot disappears.

Convergence of Fourier Series For a discontinuous function, the Fourier series converges to the average of the values around the discontinuity.

Types of convergence: Pointwise $u_n \to u$ converges pointwise for each $x \in I$ if $|u_n(x) - u(x)| \to 0$ as $n \to \infty$.

Types of convergence: Uniform $u_n \to u$ converges uniformly for all $x \in I$ if $\max_{x\in I} |u_n(x) - u(x)| \to 0, n \to \infty$, the maximum error goes to zero.

Types of convergence: mean $u_n(x) \to u(x)$ converges in the mean over I if $\int_I |u_n(x) - u(x)|dx \to 0$ as $n \to \infty$.

Strong to weak Uniform convergence implies pointwise convergence. Pointwise convergence implies mean convergence.

Examples $u_n(x) = \frac{x^2}{n}, x \in \mathbb{R}$ converges pointwise but not uniformly.

Examples $u_n(x) = x^n$ converges pointwise to 0 if $x \in [0, 1)$ and to 1 if $x = 1$ as $n \to \infty$ but does not converge uniformly.

Examples $u_n(x) = \frac{x^2}{n}, x \in [0, 1]$ over the finite bounded domain converges uniformly to zero as $n \to \infty$.

Fourier Convergence Theorem If u, u' are sectionally continuous on $[-\pi, \pi]$ and is periodic $u(-\pi) = u(\pi)$, then $u(x)$ can be written as its Fourier series when it is continuous, and if $u(x)$ is not continuous, then it converges to the average of the values around the discontinuity.

5.3 29 Apr 2015 Recitation

Green's Third Theorem

$$\iint_D \nabla^2 G u dx_0 = \iint_D G\nabla^2 u dx_0 + \int_{\partial D} (u\nabla G - G\nabla u)\cdot n dS$$

where G is a function of $\vec{x}$ and $\vec{x}_0$ such that:

$$\nabla^2_{\vec{x}_0} G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0)$$

so substituting this into the LHS of Green's Third Theorem we obtain:

$$\iint_D \nabla^2 G u(\vec{x}_0)d\vec{x}_0 = \iint_D \delta(\vec{x} - \vec{x}_0)u(\vec{x}_0)d\vec{x}_0 = u(\vec{x})$$

and hence:

$$\begin{aligned} u(\vec{x}) &= \iint_D G\nabla^2 u(\vec{x}_0)d\vec{x}_0 + \int_{\partial D} (u(\vec{x}_0)\nabla G - G\nabla u(\vec{x}_0))\cdot n dS \\ &= \iint_D G f(\vec{x}_0)d\vec{x}_0 + \int_{\partial D} (u(\vec{x}_0)\nabla G - G\nabla u(\vec{x}_0))\cdot n dS \end{aligned}$$

If we have Dirichlet boundary conditions, then we will like to pick $G = 0$ on the boundary. Then we just need to find the gradient of the Green's function.

Method of Images Consider the Free Space Green's function (using the positive $\Delta = \nabla^2$):

$$G(\vec{x}, \vec{x}_0) = \begin{cases} \frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0|, & n = 2 \\ -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}_0|}, & n = 3 \end{cases}$$

which satisfies:

$$\nabla^2 G = \delta(\vec{x} - \vec{x}_0)$$

on all of $\mathbb{R}^2, \mathbb{R}^3$.

Neumann Boundary Conditions Change the sign of the fundamental solution added to form the Green's function (for linear boundaries)

5.4 Friday 1 May 2015

Definition: Sectionally continuous $u \in S.C.^0([a,b])$ if $u \in C^0$ in a finite number of subintervals $(a, x_1), \ldots, (x_{n-1}, b)$ and $u(x_j + 0), u(x_j - 0)$ exist for all $j = 1, \ldots, n-1$.

Fourier Theorem (previously) If both a function and its derivative $u, u' \in S.C.^0([-\pi, \pi])$ then:

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

holds pointwise for all x if $u(x)$ is continuous and:

$$\frac{u(x+0) + u(x-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

for all x otherwise.

If $u \in S.C.^0([-\pi, \pi])$, we define the new function:

$$\tilde{u}(x) \equiv \frac{u(x+0) + u(x-0)}{2}, \quad \forall x \in [-\pi, \pi]$$

Now when $u(x)$ is continuous at the point, then $\tilde{u}(x) = u(x)$ clearly. If $u(x)$ is discontinuous then $\tilde{u}(x)$ is now defined at the point of discontinuities and has value equal to the average of the adjacent values. Now consider $\tilde{u}(x)$ to represent all sectionally continuous but discontinuous functions in the future.

Updated Fourier Theorem If $u, u' \in S.C.^0$, the series converges pointwise for all $x \in [-\pi, \pi]$ (using the average $\tilde{u}$ above). If $u \in C^0([-\pi, \pi]), u(\pi) = u(-\pi)$ and $u' \in S.C.^0([-\pi, \pi])$ then the series converges uniformly.

Decay of Fourier Coefficients If $u \in C^{(k)}([-\pi, \pi])$, then $|a_n|, |b_n|$ decays at least as fast as $\frac{1}{n^{k+1}}$. Intuitive understanding:

$$|a_n| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos(nx) \right| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{n^k} f(x) dx \right| \leq \frac{1}{n^k \pi}, \quad f(x) = \sin nx \text{ or } \cos nx$$

Rate of convergence Note that:

$$|u^{(k)}(x)| = \left| \sum_{n=1}^{\infty} n^k a_n f(x) + n^k b_n g(x) \right| \leq \sum_{n=1}^{\infty} n^k (|a_n| + |b_n|), \quad f, g \text{ is trigonometric}$$

Hence if $|a_n|, |b_n|$ decay as $n^{-k-(1+\epsilon)}$ for some $\epsilon > 0$ (because the infinite sum of $\frac{1}{n}$ does not converge but $n^{-(1+\epsilon)}$ does) then $u \in C^k$.

Theorem: Continuity of Fourier representations :

- Local information to global information: If $u \in C^{(k)}$ then $|a_n|, |b_n| \sim n^{-k}$.
- Global information to local information: If $|a_n|, |b_n| \sim n^{-k-(1+\epsilon)}$ then $u \in C^{(k)}$.

Integration of Fourier Series Let $V(x) = \int_0^x u(s)ds$. We will like $V(x)$ to be periodic. Hence we require $V(-\pi) = V(\pi)$ which implies that:

$$\int_0^{\pi} u(s)ds = \int_0^{-\pi} u(s)ds \implies \int_{-\pi}^{\pi} u(s)ds = 0$$

and hence u has mean zero. Then we write $u(x)$ just in terms of the trigonometric functions (without the constant term):

$$u(x) = \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$\implies V(x) = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) - \frac{b_n}{\pi} \cos(nx) + \frac{b_n}{n} = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]$$

where the constant term arises from the integration of the cosine.

where we define:

$$A_0 = 2\sum_{n=1}^{\infty} \frac{b_n}{n} = \frac{1}{\pi} \int_{-\pi}^{\infty} V(x)dx$$

$$A_n = \frac{-b_n}{n}$$

$$B_n = \frac{a_n}{n}$$

Then A_n and B_n decay to zero faster than b_n and a_n, and hence the infinite sum is convergent.

Attempt to Differentiate Fourier Series Proceeding term-by-term

$$u'(x) = \sum_{n=1}^{\infty} -na_n \sin(nx) + nb_n \cos(nx)$$

but now the coefficients go to zero slower than a_n and b_n. Hence we need to impose more conditions.

Theorem: Differentiation of Fourier Series If $u \in C^0$ and $u', u'' \in S.C.^0$, then:

$$u'(x) = \sum_{n=1}^{\infty} -na_n \sin(nx) + nb_n \cos(nx)$$

Parseval's Theorem Consider:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |u(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

This tells us if $|u(x)|^2$ is integrable (and hence finite) then the Fourier series is convergent by comparison.

L2 function space Define:

$$L_2([-\pi,\pi]) = \left\{ \left| \int_{-\pi}^{\pi} |u(x)|^2 dx < \infty \right| \right\}$$

Note that $S.C.^0([-\pi,\pi]) \subset L_2([-\pi,\pi])$.

Example 1 Define:

$$u(x) = \begin{cases} 0, & x \in [-\pi, 0] \\ x^{-1/4}, & x \in (0, \pi] \end{cases}$$

which is discontinuous, blows up at zero, but has finite squared integral. The Fourier Series exists for this function.

Hilbert Space Define:

$$H^{(P)} = \{u | u, \ldots, u^{(P)} \in L_2\}$$

the set of functions where the first P derivatives are L_2. For Laplace's equation, we will get $u \in H^{(1)}$.

Motivation The Fourier series converts a function into a series of numbers (the coefficients). This is a relationship between a continuous function and a discrete sequence.

Example: Fourier Series in PDEs Consider the problem:

$$\begin{gathered} \Delta u = 0, \quad (x,y) \in [-\pi,\pi] \times [-\pi,\pi] \\ u(x,-\pi) = g(x), \quad g \in C^0, \\ g(\pm\pi) = 0 \\ u(\pm\pi, y) = 0 \\ u(x,\pi) = 0 \end{gathered}$$

Now we expand g in its Fourier Series:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Consider the ansatz: The solution can be written as a Fourier series where the coefficients are functions of y:

$$u(x,y) = \frac{a_0(y)}{2} + \sum_{n=1}^{\infty} a_n(y) \cos(nx) + b_n(y) \sin(nx)$$

Then substituting $y = -\pi$, we have:

$$\begin{gathered} a_0(-\pi) = a_0 \\ a_n(-\pi) = a_n \\ b_n(-\pi) = b_n \end{gathered}$$

Since $u(\pi,y) = u(-\pi,y) = 0$, we know that this solution must be an odd series:

$$\begin{gathered} a_o(y) = 0 \\ a_n(y) = 0 \end{gathered}$$

Hence we have the ansatz:

$$u(x, y) = \sum_{n=1}^{\infty} b_n(y) \sin(nx)$$

and substituting into the PDE:

$$\sum_{n=1}^{\infty} -n^2 b_n(y) \sin(nx) + b_n''(y) \sin(nx) = 0$$

Now the coefficients have to vanish:

$$-n^2 b_n(y) + b_n''(y) = 0$$

with the boundary conditions for each b_n:

$$\begin{aligned} b_n(\pi) &= 0 \\ b_n(-\pi) &= b_n \end{aligned}$$

This ODE can be solved immediately:

$$\begin{aligned} b_n(y) &= \alpha e^{n(y-\pi)} + \beta e^{-n(y-\pi)} \\ \implies \alpha &= -\beta, \quad \beta = -\frac{b_n}{2\sinh(2n\pi)} \end{aligned}$$

and hence we have the solution:

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} \left[-\frac{b_n}{\sinh(2n\pi)} \sin(nx) \sinh(n(y-\pi)) \right] \\ &= \sum_{n=1}^{\infty} \gamma_n \sin(nx) \sinh(n(y-\pi)), \quad \gamma_n = -\frac{1}{\pi \sinh(2n\pi)} \int_{-\pi}^{\pi} g(x) \sin(nx) dx \end{aligned}$$

which is the exact same equation as obtained from the separation of variables.

Traveling waves Let $u(x,t) = v(x - at)$. Substitute this into the wave equation and integrate once to write v' in terms of v (note arbitrary constant). This is now an autonomous differential equation. Plot the phase portrait to determine asymptotic behavior - if bounded functions are required, you have a limited range of v values. Solve the ODE by partial fractions. Remove the absolute value in logarithms by noting the sign of each term based on the position of each term in the phase portrait.

Method of Characteristics, Fully Nonlinear Given $F(Du(\vec{x}), u(\vec{x}), \vec{x}) = 0$,

$$\dot{\vec{p}}(s) = -D_{\vec{x}}F(\vec{p}, z, \vec{x}) - D_z F(\vec{p}, z, \vec{x})\vec{p}$$
$$\dot{z}(s) = D_{\vec{p}}F(\vec{p}, z, \vec{x}) \cdot \vec{p}$$
$$\dot{\vec{x}}(s) = D_{\vec{p}}F(\vec{p}, z, \vec{x})$$

Green's Identities

$$\text{1st: } \int_\Omega u\Delta u dV = -\int_\Omega |\nabla u|^2 dV + \int_{\partial\Omega} u\frac{\partial u}{\partial N} d\sigma$$
$$\text{1st: } \int_\Omega u\Delta w dV = \int_{\partial\Omega} u\frac{\partial w}{\partial n} d\sigma - \int_\Omega (\nabla u)\cdot(\nabla w) dV$$
$$\text{2nd: } \int_\Omega (u\Delta w - w\Delta u) dV = \int_{\partial\Omega}\left(u\frac{\partial w}{\partial N} - w\frac{\partial u}{\partial N}\right) d\sigma$$

Polar Laplace's equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

n-D Polar coordinates

$$\Delta u = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\Lambda_{n-1}u = 0$$

Fundamental Solution

$$F(r) = \begin{cases} -\frac{1}{2\pi}\ln(|r|), & n = 2 \\ \frac{1}{4\pi}\frac{1}{|r|}, & n = 3 \end{cases}$$
$$-\Delta F(r) = \delta(r)$$

Mean-Value Property

$$u(\vec{p}_0) = \frac{1}{|S(\vec{p}_0,\epsilon)|}\int_{S(\vec{p}_0,\epsilon)} u(\vec{p}) d\sigma$$

Maximum principle Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose that u is continuous in $\bar{\Omega} = \Omega \cup \partial\Omega$ and harmonic in Ω. Then u takes its maximum and minimum value on $\partial\Omega$. If u is not constant, then u attains its maximum and minimum only on $\partial\Omega$.

$$\min_{\partial\Omega} u \le u \le \max_{\partial\Omega} u$$

Poisson Equation Bound Consider $\Delta u = f, u = g$. Then the bound for the solution is:

$$|u(\vec{p})| \le \max_{\partial\Omega}|g| + C\max_{\Omega}|f|$$

where C is a constant depending on the size of the domain. The larger the domain, the larger C is.

Liouville's Theorem A non-constant harmonic function in an entire domain (any dimension) cannot be bounded from above or below.

Corollary of Liouville's theorem The only bounded harmonic functions on the entire domain are constant functions.

Green's Functions (Dirichlet Boundary)

$$-\Delta_{\vec{p}}G(\vec{p},\vec{p}_0) = \delta(\vec{p}-\vec{p}_0)$$
$$G(\vec{p},\vec{p}_0) = 0, \quad \forall \vec{p} \in \partial\Omega$$
$$u(\vec{p}_0) = -\int_{\partial\Omega} g(\vec{p})\frac{\partial}{\partial N}G(\vec{p},\vec{p}_0) d\sigma$$

Method of Images: Constructing Green's Functions Write $G = F + H$, where F is the fundamental solution:

$$F(\vec{p}-\vec{p}_0) = \begin{cases} -\frac{1}{2\pi}\ln(|\vec{p}-\vec{p}_0|), & n = 2 \\ \frac{1}{4\pi}\frac{1}{|\vec{p}-\vec{p}_0|}, & n = 3 \end{cases}$$

Pick $H = -F$ on $\partial\Omega$ and with $\vec{p}_0 \notin \bar{\Omega}$ by adding or subtracting copies of the fundamental solution.

Spherical Coordinates

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi$$

Hyperbolic Expansions

$$\sinh(a \pm b) = \sinh a\cosh b \pm \cosh a \sinh b$$
$$\cosh(a \pm b) = \cosh a\cosh b \pm \sinh a \sinh b$$
$$\cosh(2a) = \sinh^2 a + \cosh^2 a = 2\sinh^2 a + 1 = 2\cosh^2 a - 1$$
$$\sinh(2a) = 2\sinh a\cosh a$$

Types of Harmoniticity

$$u \in C^2(\Omega) \text{ is } \begin{cases} \text{Harmonic if } -\Delta u = 0 \\ \text{Superharmonic if } -\Delta u \ge 0 \\ \text{Subharmonic if } -\Delta u \le 0 \end{cases}$$

Maximum principle for super and subharmonic functions If u is subharmonic, then $u \le \max_{\partial\Omega} u$. If u is superharmonic, then $u \ge \min_{\partial\Omega} u$. Hence for harmonic functions, both conditions hold.

Fourier Series

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin(nx)dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos(nx)dx$$

Pointwise convergence $u_n \to u$ converges pointwise for each $x \in I$ if $|u_n(x) - u(x)| \to 0$ as $n \to \infty$.

Uniform convergence $u_n \to u$ converges uniformly for all $x \in I$ if $\max_{x\in I} |u_n(x) - u(x)| \to 0, n \to \infty$, the maximum error goes to zero.

Mean convergence $u_n(x) \to u(x)$ converges in the mean over I if $\int_I |u_n(x) - u(x)|dx \to 0$ as $n \to \infty$.

Strong to weak Uniform convergence implies pointwise convergence. Pointwise convergence implies mean convergence.

Fourier Theorem If $u, u' \in S.C.^0$, the series converges pointwise for all $x \in [-\pi, \pi]$ (using the average $\tilde{u}$). If $u \in C^0([-\pi, \pi]), u(\pi) = u(-\pi)$ and $u' \in S.C.^0([-\pi, \pi])$ then the series converges uniformly.

Decay of Fourier Coefficients (and gap) If $u \in C^k$, then the coefficients decay like n^{-k}. If the coefficients decay like $n^{-k-1-\epsilon}$ for some $\epsilon > 0$, then $u \in C^k$.

Theorem: Differentiation of Fourier Series If $u \in C^0$ and $u', u'' \in S.C.^0$, then:

$$u'(x) = \sum_{n=1}^{\infty} -na_n \sin(nx) + nb_n \cos(nx)$$

Parseval's Theorem

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |u(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Alternative to Separation of Variables Consider the ansatz: The solution can be written as a Fourier series where the coefficients are functions of y:

$$u(x, y) = \frac{a_0(y)}{2} + \sum_{n=1}^{\infty} a_n(y)\cos(nx) + b_n(y)\sin(nx)$$

Poisson's Formula Given $\Delta u = 0, u(a, \theta) = f(\theta)$,

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\left[\frac{a^2 - r^2}{r^2 - 2ra\cos(\theta - \theta') + a^2}\right] d\theta'$$

Representation Theorem

$$-\Delta u = f, \text{ on } \Omega$$
$$u(\vec{p}_0) = -\int_\Omega [F(\vec{p} - \vec{p}_0)f(\vec{p})]\, dV + \int_{\partial\Omega}\left(F(\vec{p} - \vec{p}_0)\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial F(\vec{p} - \vec{p}_0)}{\partial N}\right) d\sigma$$

$$2D: \quad u(\vec{p}_0) = \frac{1}{2\pi}\int \ln|\vec{p} - \vec{p}_0| f(\vec{p}) dV - \frac{1}{2\pi}\int_{\partial\Omega}\left(\ln|\vec{p} - \vec{p}_0|\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial(\ln|\vec{p} - \vec{p}_0|)}{\partial N}\right) d\sigma$$
$$3D: \quad u(\vec{p}_0) = -\frac{1}{4\pi}\int \frac{f(\vec{p})}{|\vec{p} - \vec{p}_0|} dV + \frac{1}{4\pi}\int\left(\frac{1}{|\vec{p} - \vec{p}_0|}\frac{\partial u(\vec{p})}{\partial N} - u(\vec{p})\frac{\partial}{\partial N}\frac{1}{|\vec{p} - \vec{p}_0|}\right) d\sigma$$

Separation of Variables Rectangular space: Pick several new functions which satisfy exactly 1 boundary condition each and goes to zero at the other boundaries.

$$U \sim \{1, x, y, xy\}, \quad \mu = 0$$
$$U \sim \{\cos(\lambda x)\cosh(\lambda y), \cos(\lambda x)\sinh(\lambda y), \sin(\lambda x)\cosh(\lambda y), \sin(\lambda x)\sinh(\lambda y)\}, \quad \mu = -\lambda^2$$
$$U \sim \{\cosh(\lambda x)\cos(\lambda y), \cosh(\lambda x)\sin(\lambda y), \sinh(\lambda x)\cos(\lambda y), \sinh(\lambda x)\sin(\lambda y)\}, \quad \mu = \lambda^2$$

Impose the vanishing boundary conditions to find eigenvalues λ_n. Final solution is a linear superposition of the eigenfunctions. Equate superposition to initial data and apply Fourier's Theorem.

Separation of Variables in Polar Coordinates

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$
$$\Theta(\theta) \sim \begin{cases}\{1, \theta\} & \mu = 0\\ \{\cos(\sqrt{\mu}\theta), \sin(\sqrt{\mu}\theta), & \mu \neq 0\end{cases}$$
$$\text{Impose: } \Theta(\theta + 2\pi) = \Theta(\theta) \implies \Theta(\theta) \sim \begin{cases}\{1\}, & \mu = 0\\ \{\cos(n\theta), \sin(n\theta), & \sqrt{\mu} = n \in \mathbb{Z}^+, \mu \neq 0\end{cases}$$

The radial equation is Euler's ODE,

$$r^2 R'' + rR' - \mu R = 0$$
$$R(r) \sim \begin{cases} \{1, \ln r\}, & \sqrt{\mu} = n = 0 \\ \{r^n, r^{-n}\}, & \sqrt{\mu} = n \geq 1 \end{cases}$$

Hence under the separation of variables assumption, we have solutions that look like:

$$u(r, \theta) = \frac{a_0}{2} + c_0 \frac{\ln r}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] + \sum_{n=1}^{\infty} r^{-n} \left[c_n \cos(n\theta) + d_n \sin(n\theta)\right]$$

If the origin is in the domain, kick the r^{-n} and $\ln r$ terms. If the domain is not the unit disk, replace r with r/ρ, where ρ is the radius of the full disk.

Chapter 6

Week 6

6.1 Wednesday 6 May 2015

Heat Equation A time-dependent form of Laplace's equation:

$$u_t = \Delta u \quad \text{homogeneous}$$
$$u_t = \Delta u - f \quad \text{inhomogeneous}$$

Steady State When $u_t = 0$. Then we regain Laplace's or Poisson's equation:

$$0 = \Delta u \quad \text{homogeneous}$$
$$f = \Delta u \quad \text{inhomogeneous}$$

Energy Define:

$$E(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + f \cdot u \right) dV$$

Note that $E(u)$ is a functional, since it takes a function u into the reals.

Minimizing a functional The minimization of a convex functional (such as the energy) is given at a function u if

$$\frac{d}{d\epsilon} E(u + \epsilon v) \bigg|_{\epsilon=0} = 0, \forall \text{ suitable } v$$

What is suitable? For the Dirichlet problem, if $u|_{\partial\Omega}$ is given, then a suitable function v is a continuous function with $v|_{\partial\Omega} = 0$ so that we do not perturb the boundary condition. For the Neumann condition, $\frac{\partial u}{\partial N}\big|_{\partial\Omega}$ is given, so a suitable v has $\frac{\partial u}{\partial N}\big|_{\partial\Omega} = 0$.

Intuition Recall that for single-variable functions, x is a critical point of $f(x)$ if $f'(x) = 0$. In the terminology of minimization,:

$$\frac{d}{d\epsilon} f(x + \epsilon y) \bigg|_{\epsilon=0} = 0, \forall y \in \mathbb{R}$$
$$\implies f'(x + \epsilon y) \cdot y|_{\epsilon=0} = 0 \implies f'(x)y = 0, \forall y \implies f'(x) = 0$$

For multivariable functions, $\vec{x}$ is a critical point if every directional derivative is zero:

$$\frac{\partial f}{\partial \vec{w}}(\vec{x}) = 0, \forall \vec{w} \in \mathbb{R}^n$$

We hence examine:

$$\frac{\partial}{\partial\epsilon} f(\vec{x}+\epsilon\vec{y})\bigg|_{\epsilon=0} = 0, \forall y \in \mathbb{R}^n \implies \nabla f(\vec{x})\cdot\vec{y} = 0 \implies \nabla f(\vec{x}) = 0$$

Now in function spaces Since the energy function is convex, the critical point will be the minimizer. Then we examine the terms of the critical point condition:

$$\begin{aligned}
E(u+\epsilon v) &= \int_\Omega \left(\frac{1}{2}|\nabla u + \epsilon\nabla v|^2 + (u+\epsilon)\cdot f\right) dV \\
&= \int_\Omega \left(\frac{1}{2}(|\nabla u|^2 + 2\epsilon\nabla u\cdot\nabla v + \epsilon^2|\nabla v|^2) + u\cdot f + \epsilon v\cdot f\right) dV \\
&= \int_\Omega \left(\frac{1}{2}|\nabla u|^2\right) dV + \epsilon\int_\Omega \nabla u\cdot\nabla v dV + \frac{\epsilon^2}{2}\int_\Omega |\nabla v|^2 dV + \int_\Omega u\cdot f dV + \epsilon\int_\Omega v\cdot f dV
\end{aligned}$$

Now the derivative with respect to ϵ makes sense, and we can write:

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}E(u+\epsilon v) &= \int_\Omega (\nabla u\cdot\nabla v + f\cdot v) dV + \epsilon\int_\Omega |\nabla v|^2 dV \\
\implies \frac{\partial}{\partial\epsilon}E(u+\epsilon v)\bigg|_{\epsilon=0} &= \int_\Omega (\nabla u\cdot\nabla v + f\cdot v) dV
\end{aligned}$$

Now we want this to equal zero for all suitable v. Rearranging,

$$\begin{aligned}
\int_\Omega (\nabla u\cdot\nabla v + f\cdot v) dV &= \int_\Omega ((-\Delta u)v + f\cdot v) dV \text{ using Green's identities} \\
&= \int_\Omega (-\Delta u + f)\cdot v dV = 0 \\
&\implies -\Delta u + f = 0
\end{aligned}$$

because v can be varied provided it satisfies the suitable boundary conditions. Hence we see the relation between Poisson's equation and the energy:

The minimizer of the energy is the solution to Poisson's equation on the domain

$$\text{If } u \text{ minimizes } E(u) \text{ and } u|_{\partial\Omega} = g \text{ then } \begin{cases} \Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Heat Equation Problem

$$\begin{aligned}
u_t &= \Delta u - f(x) \\
u &= g \quad \text{on } \partial\Omega \\
u(x,0) &= \phi(x)
\end{aligned}$$

Now if u satisfies the heat equation, then:

$$\frac{d}{dt}E(u)(t) \leq 0$$

We show this explicitly,

$$E(u)(t) = \int_\Omega \left(\frac{1}{2}|\nabla u(x,t)|^2 + u(x,t)\cdot f(x)\right) dx$$

Taking the temporal derivative:

$$\frac{d}{dt}E(u)(t) = \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}|\nabla u(x,t)|^2 + \frac{\partial}{\partial t}u(x,t)\cdot f(x)dx$$
$$\implies \frac{d}{dt}E(u)(t) = \int_\Omega \nabla u(x,t)\cdot\nabla u_t(x,t) + f(x)u_t(x,t)dx$$

Now we know that on the boundary, there is no change of the function in time (because we have specified the boundary condition). Then we have that at $\partial\Omega$:

$$u_t(x,t) = 0 \quad \forall t \geq 0, x \in \partial\Omega$$

Hence, applying Green's identity to the first term in the temporal derivative,

$$\frac{d}{dt}E(u)(t) = \int_\Omega (-\Delta u(x,t) + f(x))u_t(x,t)dx$$

Since u solves the heat equation, $u_t = \Delta u - f$. Then we substitute this into the equation:

$$\frac{d}{dt}E(u)(t) = \int_\Omega (-\Delta u(x,t) + f(x))(\Delta u(x,t) - f(x))dx = -\int_\Omega (\Delta u(x,t) - f(x))^2 dx \leq 0$$

Hence we have shown that the energy of the solution decreases in time. It is strictly decreasing unless $\Delta u(x,t) = f(x)$ i.e. we are at steady state. Since the energy is a strictly decreasing function of time and is bounded below by zero, in the limit as $t \to \infty$, the solution approaches the steady state. The solution to the heat equation is the energy minimisation path from the initial condition to the steady state.

Homogeneous case Set $f = 0$. Then we have that:

$$E(u)(t) \leq E(\phi)$$

The energy at any later time must be less than or equal to the energy of the initial data.

Maximum Principle (Max-Min Theorem) Consider 1 dimension. Let $T > 0$, and suppose $u(x,t)$ is C^0 in $R = \{x \in [0,L], t \in [0,T]\}$. Let u solve the heat equation with some boundary conditions in the interior of the domain:

$$u_t = u_{xx}, \quad x \in (0,L), t \in (0,T)$$
$$u(x,0) = \phi(x), \quad x \in [0,L]$$
$$u(0,t) = g_1(t), u(L,t) = g_2(t) \quad \forall t > 0$$

Then u attains its maximum and minimum at the initial time $t = 0$ or sides $x = 0, L$. It cannot attain its maximum or minimum in the interior or at the top where $t = T$.

Intuition behind Maximum Principle Proceed by contradiction. Let the maximum (minimum) be obtained at some point (x_0, t_0) in the rectangle $0 < x_0 < L, 0 < t_0 \leq T$. Then $u_t(x_0,t_0) = 0$ or if $t_0 = T$ then $u_t(x_0,T) \geq 0$ as it increases towards the higher time boundary. We examine the second derivative in space. Since it is a maximum point, we have that $u_{xx}(x_0,t_0) \leq 0$. Now by the heat equation, $u_t(x_0,t_0) = u_{xx}(x_0,t_0)$. We need to remove the possibility that both the LHS and RHS are zero to obtain a contradiction. Let M be the maximum value of u on R. Then $u(x_0,t_0) = M$ where $0 < x_0 < L, 0 < t_0 \leq T$. Since this is a maximum point, its value must be larger than the maximum along the boundary:

$$\max_{t=0,x=0,L} u = M - \epsilon, \quad \epsilon > 0$$

Now define a new function (perturbing by a parabola):

$$v(x,t) = u(x,t) + \frac{\epsilon}{4L^2}(x-x_0)^2$$

Now $v(x_0,t_0) = M$ and at the boundary, $v|_{\partial R} \leq M - \epsilon + \frac{\epsilon}{4L^2}L^2 = M - \frac{3\epsilon}{4} < M$. Let the maximum of $v(x,t)$ occur at (x_1,t_1). Then we have that:

$$v_t - v_{xx} = u_t - (u_{xx} + \frac{\epsilon}{2L^2}) = u_t - u_{xx} - \frac{\epsilon}{2L^2}$$

Now since u satisfies the heat equation in the interior, we have that $u_t - u_{xx} = 0$. Hence $v_t - v_{xx} = -\frac{\epsilon}{2L^2} < 0$. Contradiction.

6.2 6 May 2015 Recitation

Separation of Variables: Example

$$\Delta u = 0 \quad r > 1$$
$$\nabla u \cdot N = g \quad r = 1 u \to 0 \quad r \to \infty$$

Proceed by separation of variables in polar coordinates:

$$u(r,\theta) = R(r)\Theta(\theta)$$

and substituting into Laplace's equation in polar coordinates:

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$
$$\implies \frac{R''(r)r^2 + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \mu$$

which gives two ODEs:

$$\Theta'' + \mu\Theta = 0$$
$$r^2R'' + rR' - \mu R = 0$$

The first equation can be solved immediately:

$$\Theta \sim \begin{cases} \{1, \theta\}, & \mu = 0 \\ \{\sin(\theta\sqrt{\mu}), \cos(\theta\sqrt{\mu})\}, & \mu \neq 0 \end{cases}$$

But to ensure that the solution is continuous, we impose the condition:

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$

which gives:

$$\sqrt{\mu} = n$$

Hence:

$$\Theta \sim \begin{cases} \{1\}, & n = 0 \\ \{\sin(n\theta), \cos(n\theta)\}, & n \neq 0 \end{cases}$$

Now the radial equation is an Euler equation. Substituting the ansatz $R = r^\alpha$, we obtain the characteristic equation:

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^\alpha = 0 \implies \alpha(\alpha-1) + \alpha - n^2 = 0 \implies \mu = \pm n$$

Now if $\mu = 0$, then we want to solve the ODE:

$$r \times z'(r) + z(r) = 0, z(r) = R'(r)$$

$$\implies \frac{z'(r)}{z(r)} = -\frac{1}{r} \implies \ln z(r) = -\ln r + c \implies z(r) = \frac{c_1}{r} \implies R(r) = c_1 \ln r + c_2$$

Hence if $\mu = 0$ then $R(r) \sim \{1, \ln r\}$. Blah blah.

Poisson's Equation on the Unit Disk Suppose we have a source term $\Delta u = f$. Then we expand f in terms of the Sturm-Liouville eigenfunctions (subject to the same conditions previously):

$$f = a_0 + \sum_{n=1}^{\infty} [f_n(r) \cos n\theta + \tilde{f}_n(r) \sin n\theta]$$

Then we also write the solution in terms of an unknown superposition of eigenfunctions:

$$u(r, \theta) = a_0(r) + \sum_{n=1}^{\infty} [a_n(r) \cos n\theta + b_n(r) \sin n\theta]$$

Plug this into the PDE and then take inner products with $1, \cos n\theta, \sin n\theta$ which will give a set of ODEs for $a_0(r), a_n(r), b_n(r)$

6.3 Friday 8 May 2015

Maximum principle Recall that for the problem:

$$\begin{gathered} u_t - u_{xx} = 0 \\ u(x, 0) = \phi(x) \\ u(0, t) = g_1(t), t \geq 0 \\ u(L, t) = g_2(t), t \geq 0 \end{gathered}$$

the maximum value of the solution $u(x, t)$ on the rectangle $[0, L] \times [0, T]$ occurs along the boundary $x = 0, L$ or $t = 0$. Then we defined the function:

$$v = u + \frac{\epsilon}{4L^2}(x - x_0)^2$$

and the PDE equation for v has that:

$$v_t(x_1, t_1) - v_{xx}(x_1, t_1) < 0$$

at its maximum at (x_1, t_1). But this contradicts the statement that:

$$v_t(x_1, t_1) - v_{xx}(x_1, t_1) \geq 0$$

Well-posedness of the Dirichlet heat problem: Uniqueness Let u_1 and u_2 solve the heat equation. Define $u = u_1 - u_2$. Then u solves:

$$\begin{aligned} u_t &= u_{xx} \\ u(x,0) &= 0 \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

Then the maximum and minimum on the boundary is 0, hence we have that u must be zero over the whole domain. Hence $u_1 = u_2$ and the solution must be unique.

Well-posedness of the Dirichlet heat problem: Stability Let $u_j, j = 1, 2$ solve the heat equation with:

$$\begin{aligned} (u_j)_t &= (u_j)_{xx} \\ u_j(x,0) &= \phi_j(x) \\ u_j(0,t) &= g_1^j(t) \\ u_j(L,t) &= g_2^j(t) \end{aligned}$$

two different initial data and two different boundary data. Then the difference $u = u_1 - u_2$ solves:

$$\begin{aligned} u_t &= u_{xx} \\ u(x,0) &= \phi_1(x) - \phi_2(x) \\ u(0,t) &= g_1^1(t) - g_1^2(t) \\ u(L,t) &= g_2^1(t) - g_2^2(t) \end{aligned}$$

Then by the maximum principle,

$$\max u \leq \max_{x,t \in [0,L]\times[0,T]} \{\phi_1(x) - \phi_2(x), g_1^1(t) - g_1^2(t), g_2^1(t) - g_2^2(t)\}$$

Similarly, for the minimum,

$$\min u \geq \min_{x,t \in [0,L]\times[0,T]} \{\phi_1(x) - \phi_2(x), g_1^1(t) - g_1^2(t), g_2^1(t) - g_2^2(t)\}$$

Hence we have the bound:

$$\max |u| \leq \max_{x,t \in [0,L]\times[0,T]} \{|\phi_1(x) - \phi_2(x)|, |g_1^1(t) - g_1^2(t)|, |g_2^1(t) - g_2^2(t)|\}$$

This gives us the stability condition. The difference in the solution is given by the difference in the initial data. Hence the Dirichlet problem is well-posed.

Well-posedness of the Neumann heat equation: Uniqueness Consider the Neumann problem in 1D:

$$\begin{aligned} u_t - u_{xx} &= 0 \\ u(x,0) &= \phi(x) \\ u_x(0,t) &= g_1(t) \\ u_x(L,t) &= g_2(t) \end{aligned}$$

in n dimensions,

$$
\begin{aligned}
u_t &= \Delta u \quad \text{on } \Omega \\
\frac{\partial u}{\partial N} &= g(x,t), \quad \text{on } \partial\Omega \\
u(x,0) &= \phi(x), \quad \text{on } \Omega
\end{aligned}
$$

This problem is well-posed because of the $t = 0$ initial condition.

Mean constraint Consider the time derivative of the mean (assume everything is uniformly convergent so we an exchange operators):

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega u_t(x,t)dx = \int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} d\sigma = \int_{\partial\Omega} g(x,t)d\sigma$$

This gives us the evolution of the mean in time. Integrate both sides with respect to time and use the FTC:

$$\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx + \int_0^t \left(\int_{\partial\Omega} g(x,s)d\sigma\right) ds = \int_\Omega \phi(x)dx + \int_0^t \left(\int_{\partial\Omega} g(x,s)d\sigma\right) ds$$

Corollary of Max/Min Principle for Heat Equation If u attains a maximum or minimum in the interior of the domain, then u is a constant.

Method of characteristics fails for Heat Equation

Solution of Heat Equation using Fourier Series This is a linear problem. Hence separation of variables and Fourier series are both valid ways to solve the problem and will give the same answer.

Consider the problem:

$$
\begin{aligned}
u_t &= u_{xx} \\
u(x,0) &= \phi(x) u(0,t) = 0 \\
u(L,t) &= 0
\end{aligned}
$$

Note that the function vanishes at $x = 0, L$, hence we expect that the Fourier series terms will be odd (the series will be even if the derivatives vanish at the boundaries). Take the Fourier series (letting the constant and cosine coefficients go to zero) as:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$$

where we let the coefficients vary with time. Now $b_n(0)$ is determined by the initial data:

$$
\begin{aligned}
u(x,0) = \sum_{n=1}^{\infty} b_n(0) \sin\frac{n\pi x}{L} &= \phi(x) \\
\implies b_n(0) &= \frac{2}{L}\int_0^L \phi(x) \sin\frac{n\pi x}{L} dx
\end{aligned}
$$

Now we plug the ansatz into the PDE:

$$
\begin{aligned}
u_t(x,t) - u_{xx}(x,t) &= \sum_{n=1}^{\infty} b_n'(t) \sin\frac{n\pi x}{L} + b_n(t)\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} = 0 \\
\implies \sum_{n=1}^{\infty} &\left(\frac{n^2\pi^2}{L^2} b_n(t) + b_n'(t)\right) \sin\frac{n\pi x}{L} = 0 \\
\implies \frac{n^2\pi^2}{L^2} &b_n(t) + b_n'(t) = 0, \quad \forall n
\end{aligned}
$$

This is an ODE for b_n, which immediately has solution:

$$b_n(t) = b_n(0)e^{-tn^2\pi^2/L^2}$$

Hence the solution is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(0)e^{-\frac{n^2\pi^2}{L^2}t} \sin\frac{n\pi x}{L}$$

We further define:

$$\beta_n \equiv b_n(0) = \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}dx$$

so that we can write the series as:

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n e^{-\frac{n^2\pi^2}{L^2}t} \sin\frac{n\pi x}{L}$$

Theorem 1: Convergence of Series If $\phi \in C^0$ and $\phi' \in S.C.$ then the series converges uniformly.

Theorem 2: Heat Equation Solutions are infinitely smooth regardless of initial data If $\phi \in S.C.$ then the solution to the heat equation $u \in C^\infty$ for all $x \in [0, L], t > 0$. All discontinuities in the initial data vanish immediately.

Proof of Theorem 2 Take k derivatives in space of the series solution:

$$\left|\frac{\partial^k u(x,t)}{\partial x^k}\right| \leq \left|\sum_{n=1}^{\infty} \beta_N e^{-\frac{n^2\pi^2}{L^2}t} \frac{n^k\pi^k}{L^k} \sin\frac{n\pi x}{L}\right| \quad \text{which could be cosine too}$$

and the Fourier coefficients of the k-th derivative also decrease exponentially hence the series converges.

Note that this arises because the original coefficients decrease exponentially, which is faster than any polynomial decrease. Hence $u \in C^\infty$.

At steady state In the limit that $t \to \infty$, note that all the exponential coefficients vanish, hence the solution is zero. This is expected because the steady state problem becomes:

$$u_{xx} = 0$$
$$u(0) = u(L) = 0$$

Example: Triangle wave initial data Consider:

$$\phi(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ 2 - 2x & x \in [\frac{1}{2}, \frac{3}{2}] \\ 2x - 4 & x \in [\frac{3}{2}, 2] \end{cases}$$

which has Fourier series:

$$\phi(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\frac{n\pi x}{L}$$

The solution is hence the same Fourier series with the coefficients containing exponentials:

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^2}e^{-\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi x}{L}$$

Backward time heat equation Given the problem:

$$\begin{gathered} u_t = u_{xx}, \quad -\epsilon \le t \le 0 \\ u(x,0) = \phi(x) \\ u(0,t) = u(L,t) = 0, \quad -\epsilon \le t \le 0 \end{gathered}$$

for some positive ϵ. Note that we can introduce the change of variables to consider negative time. Then we have the problem:

$$u_t = -u_{xx}$$

for the same boundary conditions. Suppose that we could solve this problem. Then we will be able to obtain the solution for time equals $-\epsilon$, which we call ψ:

$$\psi(x) = u(x,-\epsilon)$$

Now we can consider the heat equation starting from $t = -\epsilon$:

$$\begin{gathered} v_t = v_{xx}, \quad -\epsilon \le t \le 0 \\ v(x,-\epsilon) = \psi(x) \\ v(0,t) = v(L,t) = 0 \quad -\epsilon \le t \le 0 \end{gathered}$$

By uniqueness, v and u must have the same solution. But if $\psi \in S.C.$, then the forward heat equation states that $v(x,t) \in C^\infty, \forall t \ge -\epsilon$. But this implies that $u(x,0) \in C^\infty$ also. But we assumed that we could start with any $S.C.$ data for the u problem. Hence there exists no $\psi(x)$ such that the value of $v(x,0) = \phi(x)$ if $\phi(x) \notin C^\infty$. This problem is not stable with respect to initial parameters, because the initial data must be C^∞ to have a solution.

Chapter 7

Week 7

7.1 Monday 12 May 2015

Dirichlet Heat Equation Solution Recall that the solution of:

$$\begin{aligned} u_t = u_{xx}, \quad \forall t \geq 0, x \in (0, L) \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \forall t \geq 0 \\ u(x,0) = \phi(x), \quad \forall x \in [0,L], t = 0 \end{aligned}$$

is given by:

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \beta_n e^{-n^2\pi^2 t/L^2} \sin \frac{n\pi x}{L} \\ \beta_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx \end{aligned}$$

Neumann Heat Equation Solution For the Neumann boundary conditions,

$$\begin{aligned} u_t = u_{xx} \\ u_x(0,t) = 0 \quad u_x(L,t) = 0 \quad \forall t \geq 0 \\ u(x,0) = \phi(x), \quad \forall x \in [0,L], t = 0 \end{aligned}$$

was given by the even series:

$$\begin{aligned} u(x,t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 t/L^2} \cos \frac{n\pi x}{L} \\ \alpha_n = \frac{2}{L} \int_0^L \phi(x) \cos \frac{n\pi x}{L} dx \end{aligned}$$

Heat Equation in N dimensions Given:

$$\begin{aligned} u_t = \Delta u, \quad t > 0, \vec{x} \in \Omega \\ u(\vec{x},t) = 0, \quad \vec{x} \in \partial\Omega, t > 0 \\ u(\vec{x},0) = \phi(\vec{x}), \quad \vec{x} \in \Omega, t = 0 \end{aligned}$$

Procedure for Solution :

- Find the eigenfunction and eigenvalue solutions to the problem $-\Delta v_n(\vec{x}) = \lambda_n v_n(\vec{x}), \lambda_n \geq 0$.

- Write the solution in the basis of eigenfunctions.

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n t} v_n(x)$$

- Substitute into PDE.

$$\begin{aligned}
0 = u_t - u_{xx} &= \sum_{n=1}^{\infty} \left(-c_n \alpha_n e^{-\alpha_n t} v_n(x) - c_n e^{-\alpha_n t} \Delta v_n(x)\right) \\
&= \sum_{n=1}^{\infty} \left(-c_n \alpha_n e^{-\alpha_n t} v_n(x) + c_n e^{-\alpha_n t} \lambda_n v_n(x)\right) \\
&= \sum_{n=1}^{\infty} c_n v_n(x) e^{-\alpha_n t} \left(-\alpha_n + \lambda_n\right)
\end{aligned}$$

Hence we require that

$$\alpha_n = \lambda_n$$

Now substitute the ansatz into the initial data:

$$\begin{aligned}
\phi(\vec{x}) &= \sum_{n=1}^{\infty} c_n v_n(\vec{x}) \\
\implies \int_\Omega v_k(\vec{x}) \phi(\vec{x}) d\vec{x} &= \sum_{n=1}^{\infty} c_n v_k(\vec{x}) v_n(\vec{x}) d\vec{x} = c_k
\end{aligned}$$

where we assume that the eigenfunctions have been chosen to be normalized. $\int_\Omega |v_n(\vec{x})|^2 d\vec{x} = 1$. Then the solution is:

$$\begin{aligned}
u(\vec{x},t) &= \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} v_n(\vec{x}) \\
c_n &= \frac{1}{\int_\Omega |v_n(\vec{x})|^2 d\vec{x}} \int_\Omega \phi(\vec{x}) v_n(\vec{x}) d\vec{x}
\end{aligned}$$

Example Let u solve the 3D heat equation:

$$\begin{aligned}
u_t = \Delta u \quad \vec{x} \in B(0,1) \\
u(\vec{x},t) = 0 \quad \vec{x} \in S(0,1) \\
u(\vec{x},0) = T_0 \quad t = 0
\end{aligned}$$

The eigenvalue problem is:

$$-\Delta v_n = \lambda_n v_n, \vec{x} \in B(0,1)$$

Proceed using spherical coordinates. Then we want to solve:

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_n}{\partial r}\right) + \lambda_n v_n = 0$$

Let $v_n(r) = \frac{f_n(r)}{r}$. Then we want:

$$
\begin{gathered}
\frac{\partial}{\partial r}\left(r^2\left(\frac{f_n'(r)}{r}-\frac{f_n(r)}{r^2}\right)\right)+r\lambda_n f_n(r)\\
\implies f_n'(r)+rf_n''(r)-f_n'(r)+\lambda_n r f_n(r)=0\\
\implies f_n''(r)+\lambda_n f_n(r)=0\\
\implies f_n(r)=c_1\sin n\pi r+c_2\cos n\pi r,\quad \lambda_n=n^2\pi^2
\end{gathered}
$$

We reject the cosine solution so that the function is bounded in the domain, which includes the origin. Hence we write the eigenfunctions as:

$$
v_n(r)=\frac{\sin(n\pi r)}{r}
$$

and the general solution to the heat equation will be:

$$
u(x,t)=\sum_{n=1}^{\infty}c_n e^{-n^2\pi^2 t}\frac{\sin n\pi r}{r}
$$

Substituting into the initial conditions,

$$
\begin{gathered}
T_0=\sum_{n=1}^{\infty}c_n\frac{\sin n\pi r}{r}\\
\implies c_n=\frac{\int_0^1\frac{\sin n\pi r}{r}T_o r^2 dr}{\int_0^1\sin^2 n\pi r dr}=2T_0\frac{(-1)^{n+1}}{\pi r}
\end{gathered}
$$

Hence the solution is:

$$
u(r,t)=\frac{2T_0}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n^2\pi^2 t}\frac{\sin n\pi r}{nr}
$$

Solving Inhomogeneous Heat Equation Consider:

$$
\begin{gathered}
u_t=u_{xx}-f(x)\\
u(0,t)=u(L,t)=0\\
u(x,0)=\phi(x)
\end{gathered}
$$

Remove the source. Then let $u^*(x)$ solve:

$$
u^*_{xx}=f(x)
$$

and define:

$$
\begin{gathered}
w(x,t)=u(x,t)-u^*(x)\\
\implies w_t-w_{xx}=u_t-u_{xx}+u^*_{xx}=-f(x)+f(x)=0
\end{gathered}
$$

Hence w solves:

$$
\begin{gathered}
w_t=w_{xx}\\
w(0,t)=w(L,t)=0\\
w(x,0)=\phi(x)-u^*(x)
\end{gathered}
$$

Time dependent forcing Consider

$$u_t = u_{xx} - f(x,t)$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = \phi(x)$$

Then make the ansatz:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}$$

and expand the force term in a series as well:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{L}$$

Substitute into the PDE to get the ODE system:

$$b_n'(t) = -\frac{n^2\pi^2}{L^2} b_n(t) - f_n(t)$$

Solve by integrating factor to get: $b_n(t)$ and substitute back into the ansatz to get the full solution.

$$u(x,t) = \int_0^t \left[\sum_{n=1}^{\infty} \left(-e^{n^2\pi^2(\tau - t)/L^2} f_n(\tau) \sin \frac{n\pi x}{L} \right) \right] d\tau = \int_0^t v(x,t;\tau) d\tau$$

Note that the $v(x,t;\tau)$ functions also satisfy the heat equation, but for a different time:

$$v_t = v_{xx}, t > \tau$$
$$v(0,t;\tau) = v(L,t;\tau) = 0, \quad t > \tau$$
$$v(x,\tau,\tau) = f(x,\tau), \quad t = \tau$$

Think of v as impulses that carry the information from one timestep. We add up the impulses from time zero to time t. This is known as Duhamel's principle: If $v(x,t;\tau)$ solves the above heat equation, then the general solution solves the full heat equation. We can translate inhomogeneous problems into integrals of the solutions to homogeneous problems.

$$u(x,t) = \int_0^t v(x,t;\tau) d\tau$$

7.2 Wednesday 13 May 2015

Quiz 3 material (Heat Equation) Chapter 9 Sections 1-4

Fundamental Solution of the Heat Equation over all space Let u solve:

$$u_t = u_{xx}, \quad x \in \mathbb{R}, t > 0$$
$$u(x,0) = \phi(x), \quad x \in \mathbb{R}, t = 0$$

with boundary conditions such that:

$$|u(x,t)|, |u_x(x,t)| \to 0, \quad \text{as } |x| \to \infty$$

Recall that the fundamental solution for Laplace equation had Laplacian equal to the delta function. We take a different approach here using a scaling argument.

Consider $a > 0$. Then the substitution:

$$x \to ax = x'$$
$$t \to a^2 t = t'$$

results in the modified equation:

$$\frac{\partial}{\partial t} u(x',t') = u_{t'}(x',t')a^2$$
$$\frac{\partial^2}{\partial x^2} = u_{x'x'}(x',t')a^2$$
$$\implies \left(\frac{\partial}{\partial t'} - \frac{\partial^2}{\partial (x')^2}\right) u(x',t') = 0$$

and the change of variables also solves the heat equation. If $u(x,t)$ solves the heat equation, then $u(ax, a^2t)$ also solves the heat equation.

Pick the scaling factor $a = \frac{1}{\sqrt{t}}$. Then write:

$$u(x',t') = u(ax, a^2t) = u\left(\frac{x}{\sqrt{t}}, 1\right)$$

This suggests that we should look for solutions of the form v:

$$v\left(\frac{x}{\sqrt{t}}\right) = u\left(\frac{x}{\sqrt{t}}, 1\right)$$

Let the initial data be integrable.

$$\left|\int_{\mathbb{R}} \phi(x)dx\right| < \infty$$

and let $|u|, |u_x| \to 0$ as $|x| \to \infty$. Examine the rate of change of the mean of the solution:

$$\frac{d}{dt}Mean(u) = \int_{\mathbb{R}} u_t(x,t)dx = \int_{\mathbb{R}} u_{xx}dx = 0$$

which equals to zero by considering the boundary conditions at infinity. Now we have that the mean is a constant for all $t > 0$. Hence we want the solution v to have constant mean too:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} v\left(\frac{x}{\sqrt{t}}\right) dx = \text{constant in time}$$
$$\implies \int_{-\infty}^{\infty} v(y)dy = \text{constant in time}$$

after making the change of variables to y. Hence we examine solutions of the form:

$$\frac{1}{\sqrt{t}} v\left(\frac{x}{\sqrt{t}}\right)$$

Lemma If $v \in C^0$ and $|\int v(y)dy| < \infty$ then we define $v_\epsilon(y) = \frac{1}{\epsilon}v(y/\epsilon)$ that limits to a delta function as $\epsilon \to 0$.

Hence observe that

$$\frac{1}{\sqrt{t}}v\left(\frac{x}{\sqrt{t}}\right)$$

limits to a delta function as $t \to 0$. Substitute this ansatz into the PDE:

$$\begin{aligned}
\frac{\partial}{\partial t}\left[t^{-1/2}v(xt^{-1/2})\right] &= -\frac{1}{2}t^{-3/2}v(xt^{-1/2}) - \frac{1}{2}t^{-3/2}\frac{x}{\sqrt{t}}v'(xt^{-1/2}) \\
\frac{\partial^2}{\partial x^2}\left[t^{-1/2}v(xt^{-1/2})\right] &= t^{-3/2}v''(xt^{-1/2}) \\
\implies \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\frac{1}{\sqrt{t}}v\left(\frac{x}{\sqrt{t}}\right) &= -\frac{1}{2}t^{-3/2}\left(v\left(\frac{x}{\sqrt{t}}\right) + \frac{x}{\sqrt{t}}v'\left(\frac{x}{\sqrt{t}}\right)\right) - t^{-3/2}v''\left(\frac{x}{\sqrt{t}}\right) = 0 \\
\implies v''(y) + \frac{1}{2}(v(y) + yv'(y)) &= 0, \quad y = \frac{x}{\sqrt{t}} \\
\implies v''(y) + \frac{1}{2}(yv(y))' &= 0 \\
\implies v'(y) + \frac{1}{2}yv(y) &= c
\end{aligned}$$

Pick $c = 0$ because we just need to find one solution. Then we want to solve:

$$\begin{aligned}
v'(y) + \frac{1}{2}yv(y) &= 0 \\
\implies v(y) &= Ae^{-y^2/4}
\end{aligned}$$

which is a Gaussian. Returning to explicit dependence on space and time:

$$\frac{1}{\sqrt{t}}v\left(\frac{x}{\sqrt{t}}\right) = \frac{A}{\sqrt{t}}e^{-x^2/4t}$$

We will like to normalize this by picking A so that it will limit to the usual delta function under $t \to 0$:

$$A = \frac{1}{2\sqrt{\pi}} \implies F(x,t) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}$$

which is the 1D fundamental solution to the Heat Equation. The general solution can be written in terms of this fundamental solution using the Green's formulation:

$$u(x,t) = \begin{cases} \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{-(x-s)^2/4t}\phi(s)ds, & t > 0 \\ \phi(x), & t = 0 \end{cases}$$

where we specifically define the function at $t = 0$ because the delta function is not well-defined at $t = 0$.

Heat Equation Fundamental Solution in higher dimensions The fundamental solution in d dimensions is just the product of the fundamental solution in 1D taken d times:

$$u(\vec{x},t) = \left(\frac{1}{2\sqrt{\pi t}}\right)^d \int_{\mathbb{R}^d} e^{-|\vec{x}-\vec{s}|^2/4t}\phi(\vec{s})d\vec{s}$$

Maximum Principle Let the solution $u \in C^0$ and be bounded for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Let $M = \sup_{x\in\mathbb{R}} \phi(x)$ and $m = \inf_{x\in\mathbb{R}} \phi(x)$ (note that we use the supremum and infimum because you do not need to actually attain the maximum and minimum values). Then:

$$m \leq u(x,t) \leq M$$

Restricted Uniqueness Condition There is at most 1 continuous and bounded solution to the heat equation over $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Example: Unbounded initial data Consider $\phi = e^{x^2/4}$. Then the solution is:

$$u(x,t) = \frac{1}{\sqrt{1-t}} e^{x^2/4(1-t)}$$

which blows up in finite time.

Informal Growing with giants condition If u grows slower than e^{Ax^2} then everything is OK. Note that this is essential in considering the convergence of the integral in the Green's formulation.

Positivity using the Maximum Principle The Heat Equation solution preserves the sign of the initial data. If $\phi(x) \geq 0 \implies u(x,t) > 0$ if $\phi(x) \neq 0, \forall x \in \mathbb{R}$, that is, if ϕ is not zero for some region of finite length. Note that the condition is strict on the solution. This is important because it shows that we cannot use the method of characteristics to solve this system since the solution immediately departs from zero where the initial data is zero. The information from the non-zero part of the initial data needs to be transferred immediately to the entire domain.

Proof of the positivity condition Basically use the Green's formulation to claim that the integral is positive if you are integrating a function that has a positive component for some finite length.

Example: Gaussian Initial Data Consider $\phi(x) = e^{-x^2/\sigma^2}$. The solution is:

$$u(x,t) = \frac{\sigma}{\sqrt{\sigma^2 + 4t}} e^{-x^2/(\sigma^2+4t)}, \forall t > 0$$

after completing the Gaussian square. Observe that the height decays as time progresses and the width expands.

Example: Delta Initial Data Let $\phi(x) = \delta(x)$ Then the solution is clearly just the fundamental solution:

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}$$

Note that even though the initial data was singular, the solution is still C^∞.

7.3 Wednesday Recitation 13 May 2015

Duhamel's Principle Given:

$$\begin{aligned} u_t &= u_{xx} + f(x,t) \quad \text{in } \Omega \\ u(x, t=0) &= 0 \\ u(x,t) &= 0, \quad \text{at } \partial\Omega \end{aligned}$$

Then suppose we had a function $v(x,t;\tau)$ such that v solves the heat equation for all τ with an initial condition at time τ instead that equals the forcing parameter at time τ:

$$\begin{aligned} v_t &= v_{xx} \quad \text{in } \Omega, \forall\tau \\ v(x,\tau;\tau) &= f(x,\tau) \quad \text{at time } t=\tau \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Then the claim is that the general solution can be written as an integral of v:

$$u(x,t) = \int_0^t v(x,t;\tau)d\tau$$

We observe that this occurs because:

$$\begin{aligned} u_t &= \int_0^t v_t(x,t;\tau)d\tau + v(x,t,t) \\ &= \int_0^t v_{xx}(x,t;\tau)d\tau + f(x,t) \\ &= \left(\int_0^t v(x,t;\tau)d\tau\right)_{xx} + f(x,t) \\ &= u_{xx} + f(x,t) \end{aligned}$$

and the zero boundary conditions are also satisfied.

Example Suppose u is smooth and satisfies $u_t - \Delta u = 0$ in $\mathbb{R}^n$. Then we want to show that $u(x,t;\lambda) = u(\lambda x, \lambda^2 t)$ is also a solution for all $\lambda \in \mathbb{R}$ and that $x \cdot \nabla u + 2tu_t$ is also a solution.

Proof for Part 1 Observe that:

$$\begin{aligned} \frac{\partial}{\partial t} u(\vec{x},t;\lambda) &= \lambda^2 u_t(\lambda\vec{x}, \lambda^2 t) \\ \frac{\partial}{\partial x_i} u(\vec{x},t;\lambda) &= \lambda u_{x_i}(\lambda\vec{x}, \lambda^2 t) \\ \implies \sum_i \frac{\partial^2}{\partial x_i^2} u(\vec{x},t;\lambda) &= \lambda^2 \Delta u \\ \implies u_t(\vec{x},t;\lambda) - \Delta u(\vec{x},t;\lambda) &= 0 \end{aligned}$$

Proof for Part 2 Apply the chain rule to the proposed function:

$$\frac{\partial}{\partial \lambda} u(x,t;\lambda) = \frac{\partial}{\partial \lambda} u(\lambda\vec{x}, \lambda^2 t) = \sum_i x_i u_{x_i}(\lambda\vec{x}, \lambda^2 t) + 2\lambda t u_t(\lambda x, \lambda^2 t) = \vec{x} \cdot \nabla_x u(\lambda\vec{x}, \lambda^2 t) + 2\lambda t u_t(\lambda\vec{x}, \lambda^2 t)$$

and taking $\lambda = 1$ gives us the required identity. Now we need to show that the partial derivatives can commute so that the proposed ansatz is a solution.

Duhamel's Principle on a Circle Given the problem:

$$\begin{aligned} u_t = \Delta u + f(\vec{x},t) \quad r < 1 \\ u(t=0) = 0 \\ u(r=1) = 0 \end{aligned}$$

We solve this by looking for solutions $v(r,\theta,t;\tau)$ such that:

$$\begin{aligned} v(r,\theta,\tau;\tau) = f(r,\theta,\tau) \\ v(r=1) = 0 \end{aligned}$$

7.4 Friday 15 May 2015

Classification of 2nd Order Linear PDE in 2D Consider:

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0, \quad \text{in } \Omega$$
$$a, b, c, e, d, e, f, g \in C^2(\Omega), \quad \Omega \subset \mathbb{R}^2$$

Define the principal part to be the 2nd order part of the PDE:

$$L_0(u) = au_{xx} + 2bu_{xy} + cu_{yy}, \quad a^2 + b^2 + c^2 \neq 0$$

where we have the additional condition so that all the coefficients do not vanish simultaneously and we do not have a second order system. We can write this system in matrix notation using the Hessian:

$$L_0(u) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \times \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \times H(u)$$

where the $\times$ operator refers to term-by-term multiplication and not matrix multiplication.

We examine the discriminant of the coefficient matrix:

$$-\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = b^2 - ac$$

Consider a new coordinate space with transformations:

$$\zeta = \zeta(x, y) \quad \eta = \eta(x, y)$$
$$F(x, y) = (\zeta(x, y), \eta(x, y)) \in C^1$$
$$\det(J_F) = \det \begin{pmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{pmatrix} \neq 0$$

which is invertible since the Jacobian is nonvanishing due to the inverse function theorem:

$$F^{-1}(\zeta, \eta) = (x(\zeta, \eta), y(\zeta, \eta))$$

Let u solve $L[u] = 0$. Then we can define the transformation of the system into the new coordinate space:

$$w(\zeta, \eta) = u(x(\zeta, \eta), y(\zeta, \zeta)) \iff u(x, y) = w(\zeta(x, y), \zeta(x, y))$$

We want to determine the PDE that w solves. Taking the derivatives of u and transforming them into derivatives of w in the new coordinate system,

$$u_x = w_\zeta \zeta_x + w_\eta \eta_x$$
$$u_y = w_\zeta \zeta_y + w_\eta \eta_y$$
$$u_{xx} = w_{\zeta\zeta}(\zeta_x)^2 + 2w_{\zeta\eta}\zeta_x\eta_x + w_{\eta\eta}(\eta_x)^2 + w_\zeta \zeta_{xx} + w_\eta \eta_{xx}$$
$$u_{yy} = w_{\zeta\zeta}(\zeta_y)^2 + 2w_{\zeta\eta}\zeta_y\eta_y + w_{\eta\eta}(\eta_y)^2 + w_\zeta(\zeta_{yy}) + w_\eta \eta_{yy}$$
$$u_{xy} = u_{yx} = w_{\zeta\zeta}\zeta_x\zeta_y + w_{\zeta\eta}(\zeta_x\eta_y + \eta_x\zeta_y) + w_{\eta\eta}\eta_x\eta_y + w_\zeta \zeta_{xy} + w_\eta \eta_{xy}$$

Examining the principal part of $L[w]$:

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{pmatrix} \times \begin{pmatrix} a & b \\ b & c \end{pmatrix} \times \begin{pmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{pmatrix}^T$$

Hence the determinant is:

$$\det \begin{pmatrix} A & B \\ B & C \end{pmatrix} = |J|^2 \det \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Note that since the Jacobian is non-vanishing and is squared, it does not affect the sign of the determinant. Then we have three cases:

If $\det(L_0) > 0$, call L hyperbolic (wave equation). If $\det(L_0) = 0$, call L parabolic (heat equation). If $\det(L_0) < 0$, call L elliptic (Laplace's Equation). First order equations (like conservation laws) are considered hyperbolic. We may also apply these classifications to non-linear equations are linearizing it.

Example Let $Lu = \Delta u = u_{xx} + u_{yy}$. Then the discriminant of L is $-1 \times 1 = -1 < 0$. Hence this operator is elliptic.

Note that we may informally factor the operator as:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

which has imaginary operators and hence we cannot use the method of characteristics.

Example 2: Tricomi PDE Consider $yu_{xx} + u_{yy}$. The determinant is:

$$-\det \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = -y$$

hence the nature of the operator depends on the position on the plane. On the upper plane, it is elliptic. On the x-axis it is parabolic, and in the lower plane it is hyperbolic.

Wave Equation Consider $u_{xx} - u_{yy} = 0$. Informally, note that we can factor the system into two real operators:

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

which is the product of the one-way wave equation moving in different directions.

Wave Equation Consider:

$$u_{tt} = \Delta u, \quad \forall(\vec{x}, t) \in \mathbb{R}^n \times \mathbb{R}$$

We need to assume that $|u|, |u_x| \to 0$ as $|x| \to \infty$.

Note that if $\vec{v} \in \mathbb{R}^n$ and $|\vec{v}|^2 = 1$, then $u(\vec{x}, t) = F(\vec{v} \cdot \vec{x} - t), F \in C^2$ satisfies the wave equation.

3D Wave Equation Consider the radially symmetric Laplacian:

$$u_{tt} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

We re-scale the solution :

$$v(r,t) = ru(r,t) \implies v_{tt} = v_{rr}$$

which satisfies the wave equation in 1D. Hence we know that:

$$v(r,t) = F(r-t) + G(r+t)$$

for two C^2 functions F, G is a solution. Hence the solution for u is:

$$u(r,t) = \frac{v(r,t)}{r} = \frac{F(r-t)+G(r+t)}{r}$$

Initial Value Problem Consider:

$$\begin{gathered} u_{tt} = u_{xx}, \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{gathered}$$

We know that one general form of the solution will be:

$$u(x,t) = F(x-t) + G(x+t)$$

which can be shown to satisfy the wave equation. Hence we assume that the solution has the above form, and substitute it into the boundary data:

$$\begin{gathered} \phi(x) = F(x) + G(x) \\ \psi(x) = -F'(x) + G'(x) \end{gathered}$$

Integrating the second term,

$$-F(x) + G(x) = \int_0^x \psi(s)ds - F(0) + G(0)$$

This gives a linear system for F and G which we may invert:

$$\begin{gathered} F(x) = \frac{1}{2}\left(\phi(x) - \int_0^x \psi(s)ds + F(0) - G(0)\right) \\ G(x) = \frac{1}{2}\left(\phi(x) + \int_0^x \psi(s)ds\right) \end{gathered}$$

and hence we have the general solution to the wave equation in 1D:

$$u(x,t) = \frac{1}{2}\left(\phi(x-t) + \phi(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds$$

Now we examine what happens if we try to move backward in time. Taking the limit as $t \to 0^+$:

$$u(x,t) = \frac{\phi(x-0) + \phi(x+0)}{2}$$

and hence we may have the weak condition that $\phi \in S.C^0$.

Suppose we are given a particular point (x_0, t_0) and we want to find the value of u there. Now (drawing in the characteristic lines), only the region of the x-axis from $x_0 - t_0$ to $x_0 + t_0$ really matters. We call this the domain of dependence $[x-t, x+t]$.

Theorem: Domain of Dependence Let $(\vec{x}_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$ and let Ω be a domain bounded by the cone $|\vec{x}-\vec{x}_0|^2 \leq (t-t_0)^2$ and $\{t \geq 0\}, \{t \leq T\}$ for some $T > t_0$. Suppose $u \in C^2$ and it solves the wave equation. Then:

$$\int_{B(\vec{x}_0, t_0 - T)} \left(|\nabla u|^2 + u_t^2\right)\Big|_{t=T} dx \leq \int_{B(\vec{x}_0, t_0)} \left(|\nabla u|^2 + u_t^2\right)\Big|_{t=0} dx$$

where the energy is:

$$E = \int (|\nabla u|^2 + u_t^2) dx$$

Hence we may write it as:

$$E(T) \leq E(0)$$

This can be used to prove uniqueness, because a ball containing zero energy (for the difference between solutions) initially will remain at zero energy for all time.

Chapter 8

Week 8

8.1 Monday 18 May 2015

Wave Equation Part 2 Given the problem:

$$\begin{aligned} u_{tt} &= \Delta u \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

where u has finite and compact support, that is:

$$|u(x,t)|, |\nabla u(x,t)| \to 0 \quad \text{as } |x| \to \infty$$

up to the finite time T. In other words, there is some $B(\vec{0}, R)$ such that $u = 0$ outside of $B(\vec{0}, R)$.

Definition: Energy

$$E(u) = \int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2) dx$$

If u solves the wave equation, then the energy is conserved:

$$\begin{aligned} \frac{d}{dt} E &= 2 \int_{\mathbb{R}^n} (u_t u_{tt} + \nabla u \cdot \nabla u_t) dx \\ &= 2 \int_{\mathbb{R}^n} (u_t u_{tt} - u_t \nabla u) dx \quad \text{Green's Identity} \\ &= 0, \quad u_{tt} = \nabla u \end{aligned}$$

Forming an explicit solution for the wave equation Recall that in 1D, a wave equation solution was:

$$u(x,t) = \frac{1}{2} \left(\phi(x-t) + \phi(x+t) \right) + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds$$

Note that we can write both terms in integral form:

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t} \left[\frac{1}{2} \int_{x-t}^{x+t} \phi(s) ds \right] + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds \\ &= \frac{\partial}{\partial t} \left[t \cdot P \int_{x-t}^{x+t} \phi(s) ds \right] + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds \end{aligned}$$

where we define:

$$P\int_{\Omega} g(x)dx = \frac{1}{|\Omega|}\int_{\Omega} g(x)dx$$

In 3D, we consider the problem:

$$\begin{aligned}(u_p)_{tt} &= \Delta u_p \\ u_p(x,0) &= 0 \\ (u_p)_t(x,0) &= p(x)\end{aligned}$$

Claim 1 We claim that $v = \frac{\partial u_p}{\partial t} \in C^2$ solves:

$$\begin{aligned}v_{tt} &= \Delta v \\ v(x,0) &= p(x) \\ v_t(x,0) &= 0\end{aligned}$$

Where we note that the conditions are flipped. Hence if we have the first solution, we can immediately find the second by differentiating with respect to time.

We first verify that v solves the wave equation. This is true because we can commute the derivatives because it is C^2, hence we can take the time derivative outside. Now it is also easy to see that:

$$v(x,0) = \frac{\partial}{\partial t}u_p(x,0) = p(x)$$

Hence we only have one condition to check:

$$v_t(x,0) = \frac{\partial^2}{\partial t^2}u_p(x,0) = \Delta u_p(x,0) = 0$$

because u_p satisfies the wave equation. Hence we have verified the properties of v.

Claim 2 We also claim that the C^3 solution of:

$$\begin{aligned}u_{tt} &= \Delta u \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x)\end{aligned}$$

in 3D is given by:

$$u = \frac{\partial}{\partial t}u_\phi + u_\psi$$

where u_ϕ and u_ψ are the solutions to the claim 1 wave equation with $u(x,0) = \phi$ or ψ respectively. We verify this claim: The solution clearly satisfies the wave equation because the individual components satisfy the wave equation. The initial position is:

$$\begin{aligned}u(x,0) &= \frac{\partial}{\partial t}u_\phi(x,0) + u_\psi(x,0) = \phi(x) + 0 = \phi(x) \\ u_t(x,0) &= \frac{\partial^2}{\partial t^2}u_\phi(x,0) + \frac{\partial}{\partial t}u_\psi(x,0) = 0 + \psi(x) = \psi(x)\end{aligned}$$

Hence it remains to solve the u_p problem from Claim 1.

Claim 3 Consider again the Claim 1 conditions:

$$(u_p)_{tt} = \Delta u_p$$
$$u_p(x,0) = 0$$
$$(u_p)_t(x,0) = p(x)$$

We now claim that if $p \in C^k(\mathbb{R}^3), k \geq 2$, then the solution of the Claim 1 problem is:

$$u_p(x,t) = \frac{1}{4\pi t}\int_{\partial B(x,t)} p(\vec{s})d\sigma_t$$

which is the average integral of p. Note that $d\sigma_t$ is the surface element of a sphere of radius t. Hence we may write it as $d\sigma_t = t^2 d\sigma_1$, where $d\sigma_1$ is the surface element of the sphere of radius unity.

We also change variables $\vec{s} = \vec{x} + t\cdot\vec{\alpha}, |\vec{\alpha}| = 1$. This gives us the simplified boundary:

$$u_p(\vec{x},t) = \frac{t}{4\pi}\int_{\partial B(0,1)} p(\vec{x} + t\cdot\vec{\alpha})d\sigma_1, \quad \alpha \in \partial B(0,1)$$

This looks like a mean multiplied by a time:

$$u_p(\vec{x},t) = t\cdot M(p,\partial B)$$

We explicitly find the mean in terms of the original parameters, integrating over the ball of increasing size:

$$M(p,\partial B(x,t)) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} p(\vec{s})d\sigma_t$$
$$\implies u_p(x,t) = t\cdot M(p,\partial B(x,t))$$

We verify that this solution satisfies the initial condition as $t \to 0$ from $t > 0$. Since we required that $u(x,0) = 0$, we just need to show that the mean is bounded so that t times the mean goes to zero as $t \to 0$. Then:

$$\lim_{t\to 0^+} M(p,\partial B(x,t)) = \lim_{t\to 0^+}\frac{1}{4\pi}\int_{B(0,1)} p(\vec{x}+\vec{\alpha}t)d\sigma_1 = \frac{1}{4\pi}\int_{\partial B(0,1)}\lim_{t\to 0^+} p(\vec{x}+t\vec{\alpha})d\sigma_1 = \frac{1}{4\pi}\int_{\partial B(0,1)} p(\vec{x})d\sigma_1 = p(x)$$

and clearly the mean is bounded (in fact it goes to a constant). Hence we satisfy:

$$\lim_{t\to 0^+} t\cdot M(p,\partial B) = 0$$

Now we verify that the time derivative at zero time is equal to $p(x)$. Differentiating the solution (and using the product rule),

$$\frac{\partial u_p}{\partial t} = \frac{1}{4\pi}\int_{\partial B(0,1)} p(\vec{x}+t\vec{\alpha})d\sigma_1 + \frac{t}{4\pi}\int_{\partial B(0,1)} \nabla p(\vec{x}+t\vec{\alpha})\cdot\vec{\alpha}d\sigma_1$$

Taking the limit as $t \to 0^+$, we note that the first term is the average of $p(x)$ around an arbitrarily small ball, hence it converges to $p(x)$. the second term goes to zero because the integral of the gradient over an arbitrarily small ball is bounded, and when multiplied by t, will go to zero. Hence we satisfy the second boundary condition:

$$\lim_{t\to 0^+} \frac{\partial u_p}{\partial t}(x,t) = p(x)$$

We now check the third part of Claim 1: that the time derivative of the solution satisfies a related system with the boundary conditions interchanged:

$$\frac{\partial}{\partial t} u_p = \frac{1}{4\pi} \int_{\partial B(0,1)} p(\vec{x} + t \cdot \vec{\alpha}) d\sigma_1 + \frac{t}{4\pi} \int_{\partial B(0,1)} \vec{\alpha} \cdot \nabla p(\vec{x} + t \cdot \vec{\alpha}) d\sigma_1 = \frac{u_p(x,t)}{t} + \frac{1}{4\pi t} \int_{\partial B(x,t)} \vec{\alpha} \cdot \nabla p(\vec{s}) d\sigma_t$$

where we transformed the second term back into the original parameters. Note that $\vec{\alpha}$ is an outward unit normal derivative. We hence apply the divergence theorem to write:

$$\frac{\partial}{\partial t} u_p = \frac{u_p(x,t)}{t} + \frac{1}{4\pi t} \int_{B(x,t)} \Delta p(\vec{x}) dx$$

Taking the time derivative again:

$$\frac{\partial^2}{\partial t^2} u_p = \frac{-1}{t^2} u_p(x,t) + \frac{(u_p)_t(x,t)}{t} - \frac{1}{4\pi t^2} \int_{B(x,t)} \Delta p(\vec{x}) dx + \frac{1}{4\pi t} \frac{\partial}{\partial t} \int_{B(x,t)} \Delta p(\vec{x}) dx$$

The first three terms cancel by noting the definition of the time derivative of u_p. Hence we have:

$$\frac{\partial^2}{\partial t^2} u_p = \frac{1}{4\pi t} \frac{\partial}{\partial t} \int_{B(x,t)} \Delta p(\vec{x}) dx$$

We can write the integral over the entire ball as the integral of the integral over infinitesimal shells:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u_p &= \frac{1}{4\pi t} \frac{\partial}{\partial t} \int_0^t \left(\int_{\partial B(x,\tau)} \Delta p(\vec{s}) ds \right) d\tau \\ &= \frac{1}{4\pi t} \int_{\partial B(x,t)} \Delta p(\vec{s}) ds \end{aligned}$$

Now u_p satisfies the wave equation, so $(u_p)_{tt} = \Delta u_p$, and hence:

$$\Delta u_p(x,t) = \frac{1}{4\pi t} \int_{\partial B(x,t)} \Delta p(\vec{s}) ds$$

This is true. Why?

Hence the 3D wave equation has general solution:

$$u(x,t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{\partial B(x,t)} \phi(s) d\sigma_t \right) + \frac{1}{4\pi t} \int_{\partial B(x,t)} \psi(s) d\sigma_t$$

2D Wave equation Consider a general solution:

$$u = \frac{\partial}{\partial t} u_\phi + u_\psi$$

Define a dummy coordinate to make this a 3D problem:

$$u_p = \frac{1}{4\pi t} \int_{\partial B[(x_1,x_2,0),t]} p(s_1, s_2) d\sigma_t$$

and we project this onto the 2D plane, noting that there are two contributions from the upper and lower hemispheres:

$$u_p(x_1, x_2, t) = \frac{1}{2\pi} \int_{B^2((x_1,x_2),t)} \frac{p(s_1, s_2)ds}{\sqrt{t^2 - |s - \vec{x}|^2}}$$

and instead of integrating on the boundary, we need to integrate on an entire domain.

General Wave Equation

$$u_{tt} = c(x)u_{xx}$$

Properties of PDE solutions :

- **First order equations/ Conservation Laws**: Can use Method of Characteristics. Has finite speeds. Has uniqueness with the entropy condition. Has bounded and $S.C^0$ solutions. Doesn't need maximum or comparison principles; it just needs ODE theory along the characteristics. As $t \to \infty$, solutions are sometimes bounded, sometimes they blow up - there is no guarantee of existence for all time (even if the initial data are positive and bounded). Problem is only well-posed for a short time. For initial conditions with finite support, for PDEs like Burger's equation, solution decreases like $\frac{1}{\sqrt{t}}$. Initial data cannot be given along a characteristic curve, otherwise the solution either doesn't exist or doesn't have uniqueness.
- **Laplace's Equation/Heat Equation**: C^∞ solutions. Has maximum principle. Can use energy methods to solve. Solution decays to steady state as $t \to \infty$. First order equations with an added viscosity term u_{xx} behave like heat equations and the solutions become smooth (not like first order equations). Can have shocks. Singularity on the boundary/in the domain immediately smooths out.
- **Wave Equation** (similar to Conservation Laws/First Order): Can use Method of Characteristics (similar). Has finite speeds. Has bounded, $S.C.^0$ solutions. Has comparison principle (weak form of maximum principle) in dimensions $n \leq 3$. Can use energy methods to solve.
- **General 2nd order PDE** Can be solved using Method of Images (unbounded domains) and Green's functions (unbounded domains)/Fundamental solutions (unbounded domains)/Eigenfunction Series Solution (bounded domains)/Separation of Variables (bounded domains)/Fourier Series (bounded domains). They give the same result. As $t \to \infty$, solution is bounded. Discontinuities on the boundary are preserved.

Chapter 9

Final Review

Wave Equation Energy Method Consider:

$$\begin{aligned} u_{tt} &= (c(x)u_x)_x \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

Define the energy:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} \left[(u_t)^2 + c(x)(u_x)^2\right] dx$$

Method of Characteristics Consider:

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad x \in \mathbb{R}, t \geq 0$$

$$u(x,0) = g(x) = \begin{cases} 1, & x < -1 \\ 0, & -1 < x < 0 \\ 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

We solve this using method of characteristics:

$$u_t + uu_x = 0$$

and hence the characteristics solve:

$$\begin{aligned} \dot{x}(s) &= u \quad x(s=0) = q \\ \dot{t}(s) &= 1 \quad t(s=0) = 0 \\ \dot{u}(s) &= 0 \end{aligned}$$

and the solutions are:

$$\begin{aligned} x &= us + q \\ t &= s \end{aligned}$$

Hence if we plot the characteristics, they are lines with slope equal to $\frac{1}{u}$ where u is the value at the boundary.

The physically correct rarefaction solution is given by the linear solution $u = \frac{x}{t}, 0 < x < 2t$ since we want the value of the solution to go from 0 to 2 as we cross the rarefaction region $0 < x < 2t$.

When the shock hits the rarefaction, the shock still persists, but its speed changes. Implement the RH condition for the shock velocity:

$$\dot{s}_1 = \sigma_1 = \frac{F(u_l) - F(u_r)}{u_l - u_r} = \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{s_1}{t}\right)^2}{1 - \frac{s_1}{t}}$$

Note that $u_r(x,t) = \frac{x}{t}$ since this is the solution in the rarefaction region. This gives an ODE, which we can solve to obtain s_1, the position of the shock in the rarefaction region.