Ph106a Book Notes

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Where unstated, the reference book is Analytical Mechanics (Hand and Finch). GS≡Goldstein, Classical Mechanics.

Definition of potential

$$V(\vec{r}) = -\int_O^{\vec{r}} \vec{F} \cdot d\vec{s}$$
$$\vec{F} = -\nabla V$$

Fictitious forces

$$\vec{a}_{Cor} = -2\vec{\Omega} \times \vec{v}$$
$$\vec{a}_{Cen} = -\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

Angular velocity

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{|r|^2}$$

And by 7.10, Pg 256,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Method of virtual work

- Identify constraints and define compatible displacements orthogonal to them.
- Write virtual displacements in Cartesian coordinates.
- Identify the external forces (not constraint forces) acting on each component to form the virtual work.
- Declare that the sum of all the virtual work (for all components of the system) is equal to the virtual work done by external non-constraint forces.
- Set $\delta W = \sum \dot{p} \cdot \delta \vec{r}$, equate the coefficients of each virtual displacement (independent coordinates) and solve for the equations of motion for the coordinates.

d'Alembert's Principle (1.18-19, p5)

$$\delta W - \dot{\vec{p}} \cdot \delta \vec{r} = 0$$
$$\implies \delta W = \frac{d}{dt}(\vec{p} \cdot \delta \vec{r}) - \vec{p} \cdot \delta \dot{\vec{r}}$$

Energy identity (1.24, p6)

$$\vec{p} \cdot \delta \vec{r} = \sum_i \frac{\partial T}{\partial \dot{x}_i} \delta x_i$$

where x_i is the ith independent coordinate.

Types of constraints (1.39-1.40, p11) For constraints that do not depend on the generalized velocities (i sums over all particles, N is the number of degrees of freedom):

$$\text{Scleronomic: } \vec{r}_i(q_1, \ldots, q_N)$$
$$\text{Rheonomic: } \vec{r}_i(q_1, \ldots, q_N, t)$$

Holonomic = Scleronomic ∪ Rheonomic

Types of constraints (Lecture 4, Cross notes) Holonomic constraints have N generalized coordinates such that the coordinates uniquely define the system allowed by the constraints and the N coordinates can be varied independently. The number of degrees of freedom is equal to the number of generalized coordinates.

Scleronomic (time independent) constraints and Rheonomic (time dependent) constraints are subsets of holonomic constraints.

Independence of coordinates (MIT OCW) With all coordinates fixed but one, the last coordinate still has full range of motion.

Holonomic Constraint Identity (1.44, p14) aka "cancelling the dots". Let $\vec{r}_i$ be the position of the ith particle and q_k be the kth generalized coordinate. Then:

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}$$

Generalized force (1.49, p15) The generalized force associated with the kth degree of freedom is:

$$\mathcal{F}_k = \sum_i \vec{F}_i^{nc} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \frac{\delta W}{\delta q_k}$$

where nc refers to the non-constraint forces only. $\vec{r}$ refers to the point of application of the force. Then the virtual work can be written as a sum over generalized coordinates and not objects (1.48, p15):

$$\delta W = \sum_k \mathcal{F}_k \delta q_k$$

Also, the generalized force can be obtained from the potential for a conservative system where V does not depend on the generalized velocities as (1.58, p18):

$$\mathcal{F}_k = -\frac{\partial V}{\partial q_k}$$

Rayleigh's dissipation function (GS 1.67, Pg 23) is a generalized force for friction:

$$\mathcal{F} = \frac{1}{2}\sum_{i=1}^{M}(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2)$$

but the derivative is taken with respect to velocity:

$$\vec{F}_{fric} = -\nabla_v \mathcal{F}$$
$$Q_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

Changing Frames It is alright to write the Lagrangian in a non-inertial frame, but be sure to define the kinetic and potential energies in an inertial frame first.

Derivatives of T (1.51-52, pg16-17)

$$\frac{\partial T}{\partial q_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$
$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

Generalized Equations of motion for holonomic system (1.54, p17)

$$\mathcal{F}_k = \frac{d}{dt}\left(\frac{dT}{d\dot{q}_k}\right) - \frac{dT}{dq_k}$$
$$k = 1, 2, \ldots, N$$

for the N degrees of freedom.

Euler-Lagrange Equations (1.60, p19)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$
$$L = T - V$$
$$k = 1, 2, \ldots, N$$

for N degrees of freedom

Electromagnetic Lagrangian (GS 1.63, Pg 22) In SI units:

$$L = \frac{1}{2}mv^2 - q\phi + q\vec{A} \cdot \vec{v}$$

Constant magnetic field vector potential (5.78, Pg 193):

$$\vec{A} = \frac{B}{2}(-y\hat{i} + x\hat{j}) = \frac{1}{2}\vec{B} \times \vec{r}$$

For CGS units (5.74, Pg 192):

$$L = \frac{1}{2}mv^2 - q\Phi + \frac{q}{c}\vec{v}\cdot\vec{A}$$

Hamiltonian (1.65, p21)

$$H \equiv \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

and the time derivative is (1.68, p21)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

A coordinate that is cyclic in the Lagrangian will also be absent from the Hamiltonian (GS Pg 344).

Constructing the Hamiltonian The Hamiltonian only contains the coordinate q and conjugate momentum p. Hence we need to eliminate $\dot{q}$ or $\boldsymbol{v}$ by expressing it in terms of q and p.

Canonically conjugate momentum (1.70, p22)

$$p = \frac{\partial L}{\partial \dot{q}}$$

Hamilton's canonical equations (Shankar, 2.5.12, Pg 88)

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i$$
$$\frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i$$

Eliminating cyclic/ignorable coordinates If the Lagrangian only depends on the time derivative but not the coordinate, that coordinate is cyclic/ignorable. Define the **Routhian**, which behaves like the Lagrangian (1.71, p23):

$$R = L - p\dot{q}$$

with $N-1$ degrees of freedom. For more than one cyclic coordinate, just sum their contributions (GS 8.48, Pg 348):

$$R = \sum_{i=s+1}^{n} p_i \dot{q}_i - L$$

where the cyclic coordinates are $q_{s+1}, \ldots, q_n$. Then it obeys:

$$\frac{\delta R}{\delta q_i} = 0, \quad i = 1, \ldots, s$$
$$\frac{\partial R}{\partial q_i} = 0, \frac{\partial R}{\partial p_i} = \dot{q}_i, i = s+1, \ldots, n$$

Euler's Equation (2.2, p46)

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right)$$

for a function that minimizes the functional integral:

$$I[y] = \int_{x_0}^{x_1} F\left(y, \frac{dy}{dx}, x\right) dx$$

Action (2.16, p51)

$$S[q] = \int L(q(t), \dot{q}(t), t) dt$$

Hamilton's principle: The physical path minimizes the action.

Variational Derivative (2.24, p53)

$$\frac{\delta L}{\delta q} \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

For N degrees of freedom (2.25, p54):

$$\delta S = \int \left(\sum_{k=1}^{N} \frac{\delta L}{\delta q_k} \delta q_k\right) dt$$

and if the degrees of freedom are independent and there are the same number of generalized coordinates as degrees of freedom, each variational derivative vanishes.

Using Lagrange Multipliers Define the generalized constraint force (2.39, p60):

$$\mathcal{N}_k \equiv \sum_{i=1}^{M} \vec{N}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

where $\vec{N}_i$ is the constraint force on the ith part of the system. These constraint forces can also be obtained from the variation derivative of the Lagrangian (2.41, Pg 60):

$$\frac{\delta L}{\delta q_k} = -\mathcal{N}_k$$

Intermediate calculations on Page 60

$$\frac{\delta T}{\delta q_k} = -\sum_i \left(-\nabla_i V + \vec{N}_i\right) \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$
$$= \sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_k} - \sum_i \vec{N}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$
$$\implies \frac{\delta T}{\delta q_k} = \frac{\partial V}{\partial q_k} - \mathcal{N}_k$$
$$\implies \frac{\delta T}{\delta q_k} - \frac{\partial V}{\partial q_k} = -\mathcal{N}_k$$
$$\implies \frac{\delta T}{\delta q_k} - \frac{\partial V}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_k}\right) = -\mathcal{N}_k$$
$$\text{because } \frac{\partial V}{\partial \dot{q}_k} = 0$$
$$\implies \frac{\delta (T-V)}{\delta q_k} = \frac{\delta L}{\delta q_k} = -\mathcal{N}_k$$

Method of Lagrange Multipliers (GS 2.22 Pg 46) Given C constraint equations $G_\alpha = 0, \alpha = 1, \ldots, C$, where the constraints are on positions and not velocities:

$$\frac{\delta L}{\delta q_k} + \sum_{\alpha=1}^{C} \lambda_\alpha \frac{\partial G_\alpha}{\partial q_k} = 0$$

For non-holonomic constraints on the differential motion of the form (Lecture 7, Yanbei):

$$\sum_k f_{jk} \delta q_k = 0$$

the extremization condition gives:

$$\frac{\delta L}{\delta q_k} + \sum_j \lambda_j f_{jk} = 0, \quad \forall k$$

Lagrangian near equilibrium points (3.13-14, Pg 85)

$$L = \frac{1}{2}(\dot{q}^2 - q^2), \quad \text{stable}$$
$$L = \frac{1}{2}(\dot{q}^2 + q^2), \quad \text{unstable}$$

Damped Simple Harmonic Oscillator + Quality Factor (3.29, Pg 90)

$$\ddot{q} + \frac{\dot{q}}{Q} + q = 0$$

with roots (3.31, Pg 91):

$$\alpha = \frac{i}{2Q} \pm \sqrt{1 - \frac{1}{4Q^2}}$$

Damping Cases Underdamped ($Q > 1/2$) (3.32, Pg 91):

$$q(t) \propto e^{-t/2Q} e^{\pm i\omega t}, \quad \omega = \sqrt{1 - \frac{1}{4Q^2}}$$

Overdamped $Q < 1/2$ (3.34, Pg 92):

$$q(t) = Ae^{\lambda_+ t} + Be^{\lambda_- y}$$
$$\lambda_\pm = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1}$$

Critically damped $Q = 1/2$ (Pg 92):

$$q(t) = Ae^{\lambda t} + Bte^{\lambda t},$$
$$\lambda = -\frac{1}{2Q}$$

Quality Factor and energy loss (3.39, Pg 94)

$$\frac{dE}{dt} = -\frac{E}{Q}$$

Finding the Green's Function (Pg 103) Set the homogeneous part of the ODE to $\delta(t - t')$. Set $G = \dot{G} = 0$ for $t < t'$. Require that G is continuous about $t = t'$ and integrate the ODE to obtain the discontinuity in $\dot{G}$ about that point. Use linearly independent homogeneous solutions to construct the unique Green's function.

Using the Green's Function for arbitrary forces (3.70, Pg 104) A particular solution can be written as a convolution with the Green's function:

$$q_p(t) = \int_{-\infty}^{t} F(t')G(t - t')dt'$$

where we take the upper limit of the integral to be $t' = t$ because $G(t - t') = 0$ for $t' > t$ in causal systems.

General solution for driving simple harmonic oscillator 1. Remove the external driving force. Use the initial conditions to write down the transient decay. 2. Solve for the Green's function using zero initial velocity and zero initial position. 3. Convolve the Green's function with the external force to construct the particular solution. 4. Add the transient decay and the particular solution to obtain the full solution.

Lorentzian for resonance, scaled-time system (3.83-84, Pg 109).

$$E \sim \frac{1}{(1-\omega)^2 + \frac{1}{4Q^2}}$$
$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$$

and the relative phase is (3.89, Pg 111):

$$\tan\phi(\omega) = -\frac{\frac{\omega}{Q}}{1 - \omega^2}$$

so that the steady-state response is (3.86, Pg 111):

$$q(t) = A(\omega)\cos(\omega t + \phi(\omega))$$

Lagrangian of forced harmonic oscillator (Problem 3.16d, Pg 120)

$$L = \frac{\dot{q}^2 - q^2}{2} + qF(t)$$

Formal solution to 1D problem (4.4, Pg 125) Consider the Hamiltonian $H = T + V = \frac{\dot{q}^2}{2} + V(q)$. The implicit solution is

$$t = \int_0^q \frac{dq'}{\sqrt{2(E - V(q'))}}$$

Central force Lagrangian (4.34-35, Pg 138)

$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - V(r)$$
$$p_\phi = \mu r^2 \dot{\phi}\sin^2\theta = l_z$$

Central force Energy (4.40, Pg 140)

$$E = \frac{1}{2}\mu(\dot{r})^2 + \frac{l^2}{2\mu r^2} + V(r)$$

Central Force Equations (4.41, Pg 141)

$$\mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{dV}{dr}$$

which can be parametrized by $u = 1/r$ to get (4.48-49, Pg 143):

$$E = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right] - ku$$
$$\frac{d^2u}{d\phi^2} + u = \frac{\mu k}{l^2}$$

where we used the change of independent variable (4.46, Pg 142):

$$\frac{d}{dt} \to \frac{L}{\mu}u^2\frac{d}{d\phi}$$

More generally for any central force, the DE is (GS 3.34, Pg 87):

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right)$$

Hence if you have $u(\phi)$, you can determine the form of the central force. The orbit can be solved for the Kepler case to give (4.51, Pg 143):

$$r = \frac{p}{1 + \epsilon\cos\phi}$$
$$p = \frac{l^2}{\mu k}, \quad \epsilon = pA$$

Observe that for $\phi = \frac{\pi}{2}, r = p$ is the semi-latus rectum. The energy of the orbit is (4.54, Pg 144):

$$E = \frac{k}{2p}(\epsilon^2 - 1)$$

In terms of the semimajor axis a, $E = -\frac{k}{2a}, a = \frac{p}{1-\epsilon^2}$ by equation 4.55, Pg 144. This is used to determine the eccentricity.

Ellipse parameters (Page 146)

$$a = \frac{p}{1 - \epsilon^2}$$
$$b = \frac{p}{\sqrt{1 - \epsilon^2}}$$
$$c = \frac{\epsilon p}{1 - \epsilon^2} = a\epsilon$$
$$r_{min} = \frac{p}{1 + \epsilon}$$
$$r_{max} = \frac{p}{1 - \epsilon}$$

Ellipse parameters (GS Pg 94-95)

$$\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}, \quad 3.57$$
$$a = \frac{-k}{2E}, \quad 3.61$$

Equation of ellipse (4.57, Pg 145)

$$\frac{(x - x_c)^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$x_c = -\frac{\epsilon p}{1 - \epsilon^2}$$

is an ellipse centered at $(x_c, 0)$ with the origin at one focus, and a, b are the semi-major and semi-minor axes.

Kepler's Third Law (4.61, Pg 147)

$$\tau = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2}$$

Eccentric and true anomalies (Pg 148) Eccentric anomaly $\mathcal{E}$ is measured from the origin when the vertical component of the ellipse is projected onto a circle with the semi-major axis as the radius (circumscribing the ellipse). The azimuthal angle ϕ is the true anomaly, measured from the focus.

The orbit can be written in terms of the eccentric anomaly (4.65-66, Pg 149):

$$r = a(1 - \epsilon \cos \mathcal{E})$$
$$x = a(\cos \mathcal{E} - \epsilon)$$
$$y = a\sqrt{1-\epsilon^2} \sin \mathcal{E}$$

Kepler's Equation (4.70, Pg 149)

$$T(\mathcal{E}) = \sqrt{\frac{\mu a^3}{k}}(\mathcal{E} - \epsilon \sin \mathcal{E})$$

Laplace-Runge-Lenz Vector (GS 3.82, Pg 103) For Kepler problem only.

$$\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r}$$

with magnitude (GS 3.86-87, Pg 104-105):

$$A = mk\epsilon$$
$$A^2 = m^2k^2 + 2mEl^2$$

The LRL vector is conserved in time: $\frac{d\vec{A}}{dt} = 0$.

Scattering (GS Pg 106+)

$$d\Omega = \sin\theta d\theta d\phi$$
$$\sigma(\Omega)d\Omega = \frac{\text{intensity scattered into } d\Omega}{\text{incident intensity}}$$
$$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|, \quad 3.93$$

where θ is the angle the asymptote makes with the incoming direction.

Equation of hyperbola (4.73, Pg 150)

$$\frac{(x-x_c)^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $x_c = \frac{\epsilon p}{\epsilon^2 - 1}$, note no negative sign.

Hyperbolic Orbits (4.79-81, Pg 153) For the attractive force,

$$T(\mathcal{E}) = \sqrt{\frac{\mu a^3}{k}}(\epsilon \sinh \mathcal{E} - \mathcal{E})$$
$$r = a(\epsilon \cosh \mathcal{E} - 1)$$
$$x = a(\epsilon - \cosh \mathcal{E})$$
$$y = a\sqrt{\epsilon^2 - 1} \sinh \mathcal{E}$$

with $a = \frac{p}{\epsilon^2-1}, p = \frac{l^2}{\mu|k|}$ (Pg 151). The energy is given by $E = \frac{|k|}{2a} > 0$ (Pg 151).

For the repulsive force (4.82-84, Pg 153):

$$T(\mathcal{E}) = \sqrt{\frac{\mu a^3}{k}}(\epsilon \sinh \mathcal{E} + \mathcal{E})$$
$$r = a(\epsilon \cosh \mathcal{E} + 1)$$
$$x = a(\epsilon + \cosh \mathcal{E})$$
$$y = a\sqrt{\epsilon^2 - 1} \sinh \mathcal{E}$$

Scattering Distance of closest approach (4.77, Pg 152):

$$r_{min} = \frac{p}{\epsilon - 1} = a(\epsilon + 1)$$

Scattering angle relation (4.78, Pg 152):

$$\sin\frac{\theta}{2} = \frac{1}{\epsilon}$$

Virial theorem (GS 3.29, Pg 86)

$$\bar{T} = \frac{n+1}{2}\bar{V}$$

for power-law potentials $V = ar^{n+1}$. For $n = -2$, $\bar{T} = -\frac{1}{2}\bar{V}$.

Noether's Theorem (5.14, Pg 174) Let a continuous general transformation of coordinates $Q(s,t)$ with N degrees of freedom be defined by $s_1, s_2, \ldots$. If the Lagrangian is invariant under this symmetry transformation, then the following parameters are constants:

$$I_i = \sum_{k=1}^{N} p_k \frac{dQ_k}{ds_i}\bigg|_{s_1=s_2=\ldots=0}$$

Theory of small oscillations (GS Pg 238+)

$$L = \sum_{i,j=1}^{N} \frac{T_{ij}\dot{q}_i\dot{q}_j - V_{ij}q_iq_j}{2}, \quad 6.7$$

where the subscripts denote partial derivatives. The EoM is:

$$\sum_{j=1}^{N} T_{ij}\ddot{q}_j + V_{ij}q_j = 0, \quad 6.8$$

A normal mode in matrix form is (6.14, Pg 242):

$$\boldsymbol{V}\boldsymbol{a} = \lambda \boldsymbol{T}\boldsymbol{a}$$
$$\implies \det|\boldsymbol{V} - \lambda\boldsymbol{T}| = 0$$

where $\boldsymbol{V} = V_{ij}, \boldsymbol{T} = T_{ij}$. We can define the inner product for distinct eigenfrequencies without degeneracy:

$$\langle \boldsymbol{a}_l, \boldsymbol{a}_k \rangle = \boldsymbol{a}_l^T \boldsymbol{T} \boldsymbol{a}_k = \delta_{lk}$$

where we required that the normal mode amplitudes be normalized $\boldsymbol{a}^T\boldsymbol{T}\boldsymbol{a} = 1$.

Let $\boldsymbol{A}$ be the matrix of normalized mode amplitudes. Then (6.23, 6.26, Pg 244):

$$\boldsymbol{A}^T\boldsymbol{T}\boldsymbol{A} = \boldsymbol{I}, \quad \boldsymbol{A}^T\boldsymbol{V}\boldsymbol{A} = \boldsymbol{\lambda}$$

where $\boldsymbol{\lambda}$ is a diagonal matrix of eigenfrequencies (squared). Hence $\boldsymbol{V}$ is diagonalized through a congruence transformation. The eigenfrequencies can be obtained using (6.26'):

$$\det|\boldsymbol{V} - \lambda\boldsymbol{I}| = 0$$

The normal coordinates of the motion $\boldsymbol{\zeta}$ are obtained using (6.41', Pg 251):

$$\boldsymbol{q} = \boldsymbol{A}\boldsymbol{\zeta}$$

The Lagrangian in normal coordinates is (implicit summing, 6.45, Pg 252):

$$L = \frac{1}{2}(\dot{\zeta}_k\dot{\zeta}_k - \omega_k^2\zeta_k^2)$$

with EoMs (6.46, Pg 252):

$$\ddot{\zeta}_k + \omega_k^2\zeta_k = 0$$

Legendre Transform (Pg 176) Consider a passive variable x and an active variable y. Let $A(x,y)$ be a function of these two variables. Define $z(x,y) = \frac{\partial A}{\partial y}$. Define $B(x,y,z) = yz - A(x,y)$. Then:

$$\frac{\partial B}{\partial z} = y$$
$$\frac{\partial B}{\partial x} = -\frac{\partial A}{\partial x}$$

Hamiltonian for special Lagrangian (GS Pg 339-340) Suppose the Lagrangian can be written as:

$$L = L_0(q,t) + \dot{q}_i a_i(q,t) + \dot{q}_i^2 T_i(q,t)$$
$$\implies L = L_0(q,t) + \dot{\boldsymbol{q}}^T\boldsymbol{a} + \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{T}\dot{\boldsymbol{q}}$$
$$\implies \boldsymbol{p} = \boldsymbol{T}\dot{\boldsymbol{q}} + \boldsymbol{a}$$
$$H = \frac{1}{2}(\boldsymbol{p}^T - \boldsymbol{a}^T)\boldsymbol{T}^{-1}(\boldsymbol{p} - \boldsymbol{a}) - L_0(q,t)$$

Small oscillations, Yanbei notation, Lecture 13

$$L = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{t}\dot{\boldsymbol{q}} - \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q}$$
$$\implies \boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{t}\dot{\boldsymbol{q}}$$
$$H = \dot{\boldsymbol{q}}^T\boldsymbol{p} - L = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{t}^{-1}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q}$$

and the canonical equations are:

$$\dot{\boldsymbol{q}} = \boldsymbol{t}^{-1}\boldsymbol{p}$$
$$\dot{\boldsymbol{p}} = -\boldsymbol{v}\boldsymbol{q}$$

Conditions to write H = T+V (GS, Pg 339) 1. Lagrangian is the sum of functions homogeneous in the generalized velocities of degrees 0, 1, or 2,

2. Equations defining generalized coordinates do not depend on time explicitly, 3. Forces are derivable from a conservative potential.

Hamilton's Equations of motion (5.32, Pg 181)

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$
$$\dot{p}_k = -\frac{\partial H}{\partial q_k}$$
$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Hamilton's Equations in symplectic notation (GS Pg 342-343)

$$\vec{\eta} \equiv \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \\ p_1 \\ \vdots \\ p_N \end{pmatrix}$$

$$\frac{\partial H}{\partial \vec{\eta}} \equiv \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_N} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_N} \end{pmatrix}$$

$$\boldsymbol{J} \equiv \begin{bmatrix} \boldsymbol{0}_n & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \boldsymbol{0}_n \end{bmatrix}$$

$$\implies \dot{\vec{\eta}} = \boldsymbol{J}\frac{\partial H}{\partial \vec{\eta}}$$

$\boldsymbol{J}$ is orthogonal (GS 8.38, Pg 343):

$$\boldsymbol{J}\boldsymbol{J}^T = \boldsymbol{J}^T\boldsymbol{J} = \boldsymbol{I}$$

Liouville's Theorem (Pg 186) The area of a patch of phase space is preserved as time progresses.

Continuity equation for phase space density (5.57, Pg 188)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\dot{q})}{\partial q} + \frac{\partial(\rho\dot{p})}{\partial p} = 0$$

This can be simplified as (5.61, Pg 189):

$$\frac{d\rho}{dt} + \rho\left[\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right] = 0$$

For systems that obey Hamilton's equations, the term in the square brackets vanishes. Liouville's Theorem can hence be stated as (5.60, Pg 189):

$$\frac{d\rho}{dt} = 0$$

Momentum space Lagrangian (5.63, Pg 190)

$$K(p,\dot{p},t) = L(q,\dot{q},t) - p\dot{q} - q\dot{p}$$

which is dynamically equivalent to the Lagrangian since the last two terms are a total time derivative.

Hamiltonian in rotating reference frame (5.71, Pg 191)

$$H_{\omega\neq 0} = H_{\omega=0} - \omega l_z$$

Larmor Frequency (5.80, Pg 193)

$$\omega_L = -\frac{eB}{2mc}$$

Hamiltonian in linearly accelerated frame (5.86, Pg 194)

$$H = \frac{p^2}{2m} + V(x,y,z) - \dot{\vec{R}}_0 \cdot \vec{p}$$

where $\vec{R}_0$ represents the motion of the frame.

Types of transformations Point transformation (6.1, Pg 208):

$$Q = Q(q(t),t)$$

Contact transformations (6.3, Pg 208):

$$Q = Q(q,p,t), \quad P = P(q,p,t)$$

Canonical transformation: contact transformations such that the structure of Hamilton's equations for all dynamical systems is preserved. That is, there exists a function $K(Q,P,t)$, the transformed Hamiltonian, that satisfies (9.5, Pg 370 Goldstein):

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

The simultaneously validity of Hamilton's principle of stationary action for the sets $\{p,q,H\}$ and $\{P,Q,K\}$ imply that the Lagrangian in both cases differs by a total time derivative and some scale factor (which is usually set to unity) (6.4, Pg 209 and GS 9.8, Pg 370):

$$\bar{L}(Q,\dot{Q},t) = L(q,\dot{q},t) - \frac{dF(q,Q,t)}{dt}$$

where F is called a generating function. F cannot contain $\dot{q}$ or $\dot{Q}$ but can depend on (q,p,t) or (Q,P,t) or any mixture of these coordinates since these have zero variation at the endpoints. A necessary and sufficient condition for F_1 is:

$$\frac{\partial^2 F_1}{\partial q \partial Q} \neq 0$$

To make sure that $\bar{L}$ does not depend on $\dot{Q}$ and $\dot{q}$, we want (6.9, Pg 210):

$$P = -\frac{\partial F_1}{\partial Q}$$
$$p = \frac{\partial F_1}{\partial q}$$

where $P \equiv \frac{\partial \bar{L}}{\partial \dot{Q}}$. These are actually 2N equations that determine F_1.

Canonical transformed Hamiltonian of the first type (6.12, Pg 211)

$$\bar{H}(Q,P,t) = H(q(Q,P),p(Q,P),t) + \frac{\partial F_1(q(Q,P),Q,t)}{\partial t}$$

4 types of canonical transformations (6.21,24,26, Pg 214-215)

$$F_3(p,Q,t) = F_1(q,Q,t) - qp$$
$$F_4(p,P,t) = F_3(p,Q,t) + PQ$$
$$F_2(q,P,t) = F_1(q,Q,t) + QP$$

and one more relation (GS Table 9.1, Pg 373):

$$F_4(p,P,t) = F_1(q,Q,t) + QP - qp$$

Canonical transformation derivatives (6.22,23,25,27, Pg 214-215)

$$\frac{\partial F_3}{\partial p} = -q \quad \frac{\partial F_3}{\partial Q} = -P$$
$$\frac{\partial F_4}{\partial p} = -q \quad \frac{\partial F_4}{\partial P} = Q$$
$$\frac{\partial F_2}{\partial q} = p \quad \frac{\partial F_2}{\partial P} = Q$$

Deriving Canonical Transformation equations (GS Pg 371-373) Start with GS Eq 9.11 with implicit summation:

$$p_q\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt}$$

Then F takes the following forms:

$$F = F_1(q,Q,t)$$
$$F = F_2(q,P,t) - Q_iP_i$$
$$F = F_3(p,Q,t) + q_ip_i$$
$$F = F_4(p,P,t) + q_ip_i - Q_iP_i$$

Expand the total derivative and compare coefficients of the time-derivative terms to obtain the generating function derivatives. The final expression should be (9.14c, Pg 372):

$$K = H + \frac{\partial F_n}{\partial t}$$

Special canonical transformations Identity (6.28, Pg 215):

$$F_2 = qP \implies Q = q, P = p$$

Exchanged roles (6.29, Pg 215):

$$F_1 = qQ \implies Q = p, P = -q$$

Infinitesimal (6.35, Pg 216):

$$F_2 = qP + \epsilon G(q, P)$$
$$\implies \begin{cases} Q = q + dt\frac{\partial G}{\partial P} \\ P = p - dt\frac{\partial G}{\partial q} \end{cases}$$

Direct conditions for canonicity (GS 9.48ab, Pg 381-382)

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P}$$
$$\left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$
$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P}$$
$$\left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

where the subscripts also indicate what the functions should be expressed in terms of.

Poisson Brackets (6.36, Pg 217)

$$[F, G]_{q,p} = \sum_{k=1}^{N}\left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right)$$

the Poisson bracket is invariant under canonical transformations (6.38, Pg 217):

$$[\bar{F}, \bar{G}]_{Q,P} = [F, G]_{q,p}$$

The position momentum Poisson bracket is (6.39, Pg 217):

$$[Q, P]_{Q,P} = 1$$

and this is a necessary and sufficient condition for a transformation to be canonical. For a multi-dimensional system, remember to check that the Poisson bracket is unity for each set of conjugate coordinates and momenta.

Poisson Brackets on coordinates (GS 9.69ab, Pg 388)

$$[q_j, q_k]_{q,p} = [p_j, p_k]_{q,p} = 0$$
$$[q_j, p_k]_{q,p} = \delta_{jk}$$

Poisson Bracket Identities (GS Pg 390)

$$[u, u] = 0, \quad 9.75a$$
$$[u, v] = -[v, u], \quad 9.75b$$
$$[au + bv, w] = a[u, w] + b[v, w], \quad 9.75c$$
$$[uv, w] = [u, w]v + u[v, w], \quad 9.75d$$
$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

and deduced from GS 9.126, Pg 410,

$$[\vec{F} \cdot \vec{G}, u] = \vec{F} \cdot [\vec{G}, u] + \vec{G} \cdot [\vec{F}, u]$$

Poisson Bracket as a time derivative (GS 9.94, Pg 396 and HF 6.130, Pg 244)

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

Matrix form of Poisson Bracket (GS 9.68, Pg 388)

$$[u, v]_{\boldsymbol{\eta}} = \left(\frac{\partial u}{\partial \boldsymbol{\eta}}\right)^T \boldsymbol{J}\frac{\partial v}{\partial \boldsymbol{\eta}}$$

Poisson bracket with symplectic vector (GS 9.99, Pg 398)

$$[\boldsymbol{\eta}, u] = \boldsymbol{J}\frac{\partial u}{\partial \boldsymbol{\eta}}$$

Levi-Civita Symbol and Cross Products with Einstein summation:

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk}\boldsymbol{e}_i a_j b_k$$
$$\implies (\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k$$

The product of two Levi-Civita symbols is (implicit summing over i):

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Infinitesimal Canonical transformation (GS Pg 385-386) The generating function for an ICT is (GS 9.62, Pg 385):

$$F_2 = q_i P_i + \epsilon G(q, P, t)$$

which allows us to relate the old and new coordinates (GS 9.63ab, Pg 386):

$$\delta p_j \equiv P_j - p_j = -\epsilon\frac{\partial G}{\partial q_j}$$
$$\delta q_j \equiv Q_j - q_j = \epsilon\frac{\partial G}{\partial p_j}$$

and the transformation equations can be written in matrix form:

$$\delta\boldsymbol{\eta} = \epsilon\boldsymbol{J}\frac{\partial G}{\partial \boldsymbol{\eta}}$$

so that the transformed canonical variables differ only infinitesimally from the initial coordinates (9.98, Pg 398):

$$\boldsymbol{\zeta} = \boldsymbol{\eta} + \delta\boldsymbol{\eta}$$

Using Poisson brackets, the infinitesimal displacement is (GS 9.100, Pg 399):

$$\delta\boldsymbol{\eta} = \epsilon[\boldsymbol{\eta}, G]$$

Hamilton-Jacobi Equation (6.42, Pg 219)

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0$$

where k runs from 1 to N. The solution $S = F_2(q_k, P_k, t)$ is called Hamilton's Principal Function.

Notation: Define $P_k \equiv \alpha_k$, which are constants obtained by solving the H-J equation. If separation of variables is used, α_k are the separation constants. Define $Q_k \equiv \beta_k = \frac{\partial S}{\partial P_k}$. The constants are obtained by solving the 2N implicit equations (6.44, Pg 220):

$$p_k(0) = \left.\frac{\partial S}{\partial q_k}\right|_{t=0}$$
$$\beta_k = \left.\frac{\partial S}{\partial \alpha_k}\right|_{t=0}$$

Hamilton's Characteristic Function (6.57, Pg 222) is the solution $W(q_k, \alpha_k)$ to the second form of the Hamilton-Jacobi equation:

$$H\left(q_k, \frac{\partial W}{\partial q_k}\right) = E$$

where E is a constant.

Matrix elements (9.24, 9.28, Pg 350-351)

$$v_{ij} = \left.\frac{1}{2}\frac{\partial^2 V}{\partial \phi_i \phi_j}\right|_{\phi_1 = \phi_j = 0}$$
$$t_{ij} = \left.\frac{1}{2}\frac{\partial^2 T}{\partial \dot{\phi}_i \dot{\phi}_j}\right|_{\dot{\phi}_1 = \dot{\phi}_j = 0}$$

so that the Lagrangian is (9.30, Pg 351):

$$L = \dot{\phi}^T \boldsymbol{t}\dot{\phi} - \phi^T \boldsymbol{v}\phi$$

with EoM (9.33, Pg 352):

$$\boldsymbol{t}\ddot{\phi} + \boldsymbol{v}\phi = 0$$

Normal modes can be written as $\phi(t) = \Phi e^{i(\omega t - \delta)}$ and satisfy (9.35, Pg 352):

$$(-\omega^2 \boldsymbol{t} + \boldsymbol{v})\Phi = 0$$

The necessary and sufficient condition for stability is for all the eigenvalues ω^2 to be positive. Mode inner product (9.43, Pg 355 and Pg 364):

$$(\Phi_i, \Phi_j) = \Phi_i^T \boldsymbol{t} \Phi_j = \delta_{ij}$$
$$\Phi_i^T \boldsymbol{v} \Phi_j = \omega_i^2 \delta_{ij}$$

Time evolution and initial state (9.50-52, Pg 357):

$$\phi(0) = \Re[\sum A_i \Phi_i]$$
$$\implies \begin{cases} \Re[A_i] = \Phi_i^T \boldsymbol{t} \phi(0) \\ \omega_i \Im[A_i] = \Phi_i^T \boldsymbol{t} \dot{\phi}(0) \end{cases}$$
$$\phi(t) = \Re\left(\sum A_i e^{i\omega_i t} \Phi_i\right)$$

Note that A_i is complex! Use initial velocities.

Normal coordinates Let:

$$\rho_i(t) = \Re[A_i e^{i\omega_i t}]$$

so that the solution becomes:

$$\sum \rho_i(t) \Phi_i$$

where the EoM for each normal coordinate is decoupled (9.63, Pg 359):

$$\ddot{\rho}_i + \omega_i^2 \rho_i = 0$$

Hill Equation (10.14, Pg 389)

$$\ddot{q} + a(t)\dot{q} + b(t)q = 0$$

where $a(t), b(t)$ are periodic. The most general solution is a superposition of cosine-like and sine-like terms (10.16, Pg 390):

$$\theta(t) = \theta(0)\theta_c(t) + \dot{\theta}(0)\theta_s(t)$$

Define the following constants at one full period of the periodic variation T of the coefficients:

$$A = \theta_c(T), \quad B = \theta_s(T)$$
$$C = \dot{\theta}_c(T), \quad D = \dot{\theta}_s(T)$$

and the matrix:

$$\boldsymbol{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that we can write (10.20, Pg 391):

$$\begin{pmatrix} \theta(nT) \\ \dot{\theta}(nT) \end{pmatrix} = \boldsymbol{M}^n \begin{pmatrix} \theta(0) \\ \dot{\theta}(0) \end{pmatrix}$$

By Liouville's theorem, $\det M = 1$ for systems without damping. A stable 1DoF system has $\mathrm{Tr}(\boldsymbol{M}) \le 2$. This gives complex eigenvalues for $\boldsymbol{M}$ which we can write as $\lambda = e^{i\mu T}$. The period of oscillatory motion is $T_{osc} = \frac{2\pi}{\mu}$.

Floquet theorem (10.15, Pg 390) Complex solutions of the Hill Equation can be put in the form:

$$q(t) = P(t)e^{\pm i\mu t}, \quad P(t+T) = P(t)$$

If μ is real, there is bounded motion.

Ph106a Misc Notes
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Trigonometric functions

$\sin\frac{\pi}{6}=\frac{1}{2}, \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}, \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}, \sin\frac{\pi}{2}=1.$
$\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}, \cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}, \cos\frac{\pi}{3}=\frac{1}{2}, \cos\frac{\pi}{2}=0.$
$\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}, \tan\frac{\pi}{4}=1, \tan\frac{\pi}{3}=\sqrt{3}, \tan\frac{\pi}{2}=\infty.$
More definitions and identities:

- $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$
- $\cos z=\frac{e^{iz}+e^{-iz}}{2}$
- $2\sin z_1\cos z_2=\sin(z_1+z_2)+\sin(z_1-z_2)$
- $\sin(z+\pi/2)=\cos z$
- $\sin(z-\pi/2)=-\cos z$
- $\sin(iy)=i\sinh y, \sin z=-i\sinh(iz), \sinh z=-i\sin(iz)$
- $\cos(iz)=\cosh z, \cosh(iz)=\cos z$
- $\sin z=\sin x\cosh y+i\cos x\sinh y$
- $\cos z=\cos x\cosh y-i\sin x\sinh y$
- $|\sin z|^2=\sin^2 x+\sinh^2 y, |\cos z|^2=\cos^2 x+\sinh^2 y$
- $\sin z=0 \iff z=n\pi, n\in\mathbb{Z}$
- $\cos z=0 \iff z=\pi/2+n\pi, n\in\mathbb{Z}$
- $\frac{d}{dz}\tan z=\sec^2 z$, $\frac{d}{dz}\cot z=-\csc^2 z$, $\frac{d}{dz}\sec z=\sec z\tan z$, $\frac{d}{dz}\csc z=-\csc z\cot z$
- $\cosh^2 y-\sinh^2 y=1$
- $\sinh(z_1+z_2)=\sinh z_1\cosh z_2+\cosh z_1\sinh z_2, \cosh(z_1+z_2)=\cosh z_1\cosh z_2+\sinh z_1\sinh z_2$
- $\sinh z=\sinh x\cos y+i\cosh x\sin y, \cosh z=\cosh x\cos y+i\sinh x\sin y$
- $|\sinh z|^2=\sinh^2 x+\sin^2 y, |\cosh z|^2=\sinh^2 x+\cos^2 y$
- $\frac{d}{dz}\tanh z=\text{sech}^2 z$, $\frac{d}{dz}\coth z=-\text{csch}^2 z$, $\frac{d}{dz}\text{sech} z=-\text{sech} z\tanh z$, $\frac{d}{dz}\text{csch} z=-\text{csch} z\coth z$

Inverse trigonometric functions

- $\sin^{-1} z=-i\log[iz+(1-z^2)^{1/2}]$
- $\cos^{-1} z=-i\log[z+i(1-z^2)^{1/2}]$, branch points at ±1
- $\tan^{-1} z=\frac{i}{2}\log\frac{i+z}{i-z}=\frac{1}{2i}\log\frac{1+iz}{1-iz}$, branch points at $\pm i$.
- $\frac{d}{dz}\sin^{-1} z=\frac{1}{(1-z^2)^{1/2}}$, depends on what branch square root is defined on. Branch points at ±1.
- $\frac{d}{dz}\cos^{-1} z=\frac{-1}{(1-z^2)^{1/2}}$, depends on what branch square root is defined on. Branch points at ±1
- $\frac{d}{dz}\tan^{-1} z=\frac{1}{1+z^2}$, branch points at $\pm i$
- $\sinh^{-1} z=\log[z+(z^2+1)^{1/2}]$, branch points at $\pm i$
- $\cosh^{-1} z=\log[z+(z^2-1)^{1/2}]$
- $\tanh^{-1} z=\frac{1}{2}\log\frac{1+z}{1-z}$, branch points at ±1.
- $\coth^{-1} z=\frac{1}{2}\log\frac{z+1}{z-1}$
- $\text{csch}^{-1} z=\log\left(\frac{1}{z}+(\frac{1}{z^2}+1)^{1/2}\right)$

- $\text{sech}^{-1} z = \log\left(\frac{1}{z} + (\frac{1}{z^2} - 1)^{1/2}\right)$

Useful Taylor series:

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, |z| < \infty$
- $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, |z| < \infty$
- $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, |z| < \infty$. Substitute $z \to iz$ in sine expansion, then multiply by $-i$ since $\sinh z = -i \sin(iz)$.
- $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, |z| < \infty$. Obtain from $\cosh z = \cos(iz)$.
- $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1$.
- $\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n, |z-1| < 1$.
- $\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \frac{1}{z^{n+1}} + \frac{1}{z^N} \frac{s^N}{z-s}$, note finite number of terms.
- $\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$.
- $\tan^{-1}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n-1}}{2n-1}$.

More Trigonometric Identities

- $\sin(x + \pi/2) = \cos x$
- $\sin(x - \pi/2) = -\cos x$
- $\sin(x \pm \pi) = -\sin x$
- $\cos(x + \pi/2) = -\sin x$
- $\cos(x - \pi/2) = \sin x$
- $\cos(x \pm \pi) = -\cos x$
- $\tan(x + \pi) = \tan x$
- $\tan(x \pm \pi/2) = -\cot x$
- $\cos 3x = 4\cos^3 x - 3\cos x$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
- $\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
- $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$
- $\cos a + \cos b = 2\cos\frac{a-b}{2}\cos\frac{a+b}{2}$