Contents

Chapter 1

Week 1

1.1 Friday, 15 Jan 2016

1.1.1 Fluid element

Consider a cube of length l_{region}. We require that the properties of fluid element to be simply defined and not vary across the length scale of the element. Mathematically:

$$l_{\text{region}} << l_{\text{scale}} \sim \frac{q}{|\nabla q|} \tag{1.1}$$

for a quantity q. We also require that the number of particles contained within the element is large:

$$n l_{\text{region}}^3 >> 1 \tag{1.2}$$

We also require that, for a collisional fluid, the mean free path be small compared to the characteristic scale, so that the collisions will serve to maximize the entropy and achieve a well defined pressure:

$$l_{\text{region}} >> \lambda \tag{1.3}$$

Note that for a collisionless fluid, $\lambda \geqslant l_{\text{region}}$ so that the system retains the memory of the initial velocity field and does not achieve the final velocity distribution that maximizes the entropy. In such a case, pressure is not a well-defined quantity.

1.1.2 Relationship between Lagrangian and Eulerian descriptions

Consider a scalar field $Q(\vec{r}, t)$. Consider two points $(\vec{r}, t) \to (\vec{r} + \delta\vec{r}, t + \delta t)$. Then the total derivative is:

$$\frac{DQ}{Dt} = \lim_{\delta t \to 0} \left[\frac{Q(\vec{r} + \delta\vec{r}, t + \delta t) - Q(\vec{R}, t)}{\delta t} \right] \tag{1.4}$$

We re-write the numerator by Taylor expansion:

$$\begin{aligned}
Q(\vec{r} + \delta\vec{r}, t + \delta t) - Q(\vec{r}, t) &= Q(\vec{r} + \delta\vec{r}, t + \delta t) - Q(\vec{r}, t + \delta t) + Q(\vec{r}, t + \delta t) - Q(\vec{r}, t) &(1.5)\\
&= (\nabla Q(\vec{r}, t + \delta t)) \cdot \delta\vec{r} + \left.\frac{\partial Q(\vec{r}, t)}{\partial t}\right|_{\vec{r}, t} \delta t &(1.6)\\
&= \delta\vec{r} \cdot \left[\nabla Q(\vec{r}, t) + \frac{\partial \nabla Q(\vec{r}, t)}{\partial t} \delta t + \dots \right] + \left.\frac{\partial Q(\vec{r}, t)}{\partial t}\right|_{\vec{r}, t} \delta t &(1.7)\\
&= \delta\vec{r} \cdot \nabla Q(\vec{r}, t) + \left.\frac{\partial Q(\vec{r}, t)}{\partial t}\right|_{\vec{r}, t} \delta t, \quad \text{to first order} &(1.8)
\end{aligned}$$

Hence we have the recipe for the derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla, \quad \vec{u} \equiv \frac{d\vec{r}}{dt} \tag{1.9}$$

The Lagrangian derivative is $\frac{D}{Dt}$, the Eulerian derivative is $\frac{\partial}{\partial t}$ and the convective derivative is $\vec{u} \cdot \nabla$.

1.1.3 Fluid Kinematics

- Streamlines: lines so that the tangent to the line at each point points in the direction of the local fluid velocity. In other words:

$$\frac{d\vec{r}}{ds} \times \vec{u} = 0 \tag{1.10}$$

- Particle paths: are the trajectories of the fluid particles:

$$\frac{d\vec{r}}{dt} = \vec{u}(\vec{r}, t) \tag{1.11}$$

- Streaklines: the locus of points such that these points have passed through a fixed point $\vec{r}_0$ at some earlier point in time:

$$\{\vec{r} | \exists t < t', \vec{r}(t) = \vec{r}_0\} \tag{1.12}$$

1.2 Monday 18 Jan 2016

1.2.1 Mass conservation

The rate of mass change is given by:

$$\frac{\partial}{\partial t} \int_V \rho dV \tag{1.13}$$

Provided that there are no sources and sinks of matter (which is a reasonable assumption), then the rate of change is solely due to the flow of mass across the boundary. Define inflow to be positive. Then:

$$\frac{\partial}{\partial t} \int_V \rho dV = -\int_S \rho \vec{u} \cdot d\vec{S} \tag{1.14}$$

where the negative sign comes fro the definition of the infinitesimal area vector as pointing outwards. Then the divergence theorem gives (after generalizing to all volumes):

$$\nabla \cdot (\rho \vec{u}) + \frac{\partial \rho}{\partial t} = 0 \tag{1.15}$$

which is the mass conservation relation/continuity equation in the Euler perspective. In the Lagrangian point of view, we replace the partial time derivative and obtain:

$$\nabla \cdot (\rho \vec{u}) + \frac{D\rho}{Dt} - \vec{u} \cdot \nabla \rho = 0 \implies \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \tag{1.16}$$

The definition of an **incompressible fluid** is that $\frac{D\rho}{Dt} = 0$. It is not the same as taking the partial derivative with respect to time to be vanishing. Then using the Lagrangian point of view, the divergence of a velocity field for an incompressible fluid must vanish.

1.2.2 Gas Pressure

Write the force per unit volume along any direction as:

$$F_i = \sigma_{ij} \frac{d\vec{S}_j}{|d\vec{S}_j|} \tag{1.17}$$

where σ_{ij} is the stress tensor. The diagonal terms correspond to the pressure: $\sigma_{ii} = p$.

1.2.3 Momentum equation

Consider the pressure acting on a surface element along the $\hat{n}$ direction: $-pd\vec{S} \cdot \hat{n}$. The net force due to pressure is the integral:

$$F = -\int_S p\hat{n} \cdot \vec{S} = -\int_V \nabla \cdot (p\hat{n}) dV = -\int_V \hat{n} \cdot \nabla p dV \tag{1.18}$$

because $\nabla \cdot \hat{n} = 0$ since $\hat{n}$ is a constant.

Now let gravity act on the fluid. We evaluate the rate of change of momentum per unit volume along $\hat{n}$:

$$\left(\frac{D}{Dt} \int_V \rho \vec{u} dV \right) \cdot \hat{n} = -\int_V \hat{n} \cdot \nabla p dV + \hat{n} \cdot \int_V \rho \vec{g} dV \tag{1.19}$$

Note that the signs of gravity and pressure gradient are opposite. Now we consider the limit where $V = \int dV = \delta V \to 0$. Then we replace the integrals:

$$\left(\frac{D}{Dt} (\rho \vec{u} \delta V) \right) \cdot \hat{n} = -\hat{n} \cdot \nabla p \delta V + \hat{n} \cdot \rho \vec{g} \delta V \tag{1.20}$$

Note that $\rho \delta V$ is the mass of the fluid element. Under mass conservation, we make take $\rho \delta V$ to be constant as we following the fluid element in time:

$$\rho \delta V \frac{D}{Dt} \vec{u} \cdot \hat{n} = -\hat{n} \cdot \nabla p \delta V + \hat{n} \cdot \rho \vec{g} \delta V \tag{1.21}$$

Now this equality must hold for all $\delta V, \hat{n}$, and hence we have:

$$\rho \frac{D}{Dt} \vec{u} = -\nabla p + \rho \vec{g} \tag{1.22}$$

This is the **Momentum equation in Lagrangian form**. Note further that we can write it out in **Eulerian form** in:

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \rho \vec{g} \tag{1.23}$$

1.2.4 More about the stress tensor

Consider the rate of change of momentum in Eulerian form:

$$\frac{\partial}{\partial t} (\rho u_i) = \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} \tag{1.24}$$

Substituting the momentum equation for $\rho \partial_t u_i$, we obtain:

$$\frac{\partial}{\partial t} (\rho u_i) = u_i \frac{\partial \rho}{\partial t} - \rho (u_j \partial_j) u_i - \partial_j (p \delta_{ij}) + \rho g_i \tag{1.25}$$

Now by the continuity equation, $\partial_t \rho = -\partial_j(\rho u_j)$. Hence we have:

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_j u_i + p\delta_{ij}) + \rho g_i \tag{1.26}$$

The term $\partial_j(\rho u_j u_i)$ is called the ram pressure. Note that we wrote the pressure using the Kronecker delta so that we can bring the pressure term into the partial derivative with respect to j instead of i. We hence have the terms of the stress tensor:

$$\sigma_{ij} = \rho u_j u_i + p\delta_{ij} \tag{1.27}$$

so that the momentum change is given by:

$$\frac{\partial(\rho u_i)}{\partial t} = \partial_j \sigma_{ij} + \rho g_i \tag{1.28}$$

Example Consider fluid flowing through a pipe aligned in the $\hat{y}$ direction. The diagonal terms of the stress tensor is given by the pressure. Doing an average over the fluid motion, and noting that there is a bulk flow of the motion along the $\hat{y}$ direction ,we note that we must include an additional ram pressure in the σ_{yy} term:

$$\sigma = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix} \tag{1.29}$$

All other ram pressure components $u_i u_j$ average to zero if the fluid is isotropic other than the $\hat{y}$ flow.

Example applied to astrophysics Consider a galaxy with cross section radius R_d flowing through the intercluster medium. The ram pressure is given by $p_{ram} = \rho_{ICM} v^2$. The amount of swept-up material from the ICM is also given by $\pi R_d^2 \rho_{ICM} v$. Hence we note that the momentum transfer to the galaxy (per unit cross-sectional area) from the ICM is just the mass transfer multiplied by the relative velocity v.

1.3 Wednesday 20 Jan 2016

1.3.1 Conservative force

Recall that a conservative force can be written as the gradient of a potential (scaling constant and sign arbitrary). The following are equivalent:

$$\vec{F} = \nabla \phi \tag{1.30}$$

$$\oint \vec{F} \cdot d\vec{l} = 0 \tag{1.31}$$

1.3.2 Gravitational potential

We define:

$$\vec{g} = -\nabla \psi \tag{1.32}$$

1.3.3 Poisson Equation

For continuous systems,

$$\vec{g}(\vec{r}) = -\int_V \frac{G\rho(\vec{r'})(\vec{r} - \vec{r'})d^3\vec{r'}}{|\vec{r} - \vec{r'}|^3} \tag{1.33}$$

First consider taking the divergence of an inverse square field:

$$\nabla_r \left(\frac{\vec{r}}{|r|^3} \right) \tag{1.34}$$

Integrating both sides over a vanishing volume and noting that the integral is constant, we note that:

$$\nabla_r \left(\frac{\vec{r}}{|r|^3} \right) = 4\pi\delta(\vec{r}) \tag{1.35}$$

Hence we may take the derivative of both sides of the gravitational field equation:

$$\nabla_r \cdot \vec{g} = -4\pi G \int \rho(\vec{r'})\delta(\vec{r} - \vec{r'})d^3\vec{r'} = -4\pi G\rho(\vec{r}) \tag{1.36}$$

Using the definition of the gravitational potential, we hence obtain Poisson's equation:

$$\nabla^2\psi + 4\pi G\rho(\vec{r}) = 0 \tag{1.37}$$

In integral form:

$$\int_S \vec{g} \cdot d\vec{S} = -4\pi G M_{enc} \tag{1.38}$$

Chapter 2

Week 2

2.1 Friday 22 Jan 2016

2.1.1 Gravitational binding energy

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_{i}\frac{GM_iM_j}{|\vec{r}_j - \vec{r}_i|} \tag{2.1}$$

For a continuous system:

$$\Omega = \frac{1}{2}\int dV \rho(\vec{r})\psi(\vec{r}) = -G\int_0^\infty \frac{M(r)dm}{r} \tag{2.2}$$

2.1.2 Virial theorem

Consider the second derivative of the moment of inertia:

$$\frac{1}{2}\frac{d^2I}{dt^2} = \sum_i \vec{r}_i \cdot \vec{F}_i + 2T \tag{2.3}$$

where T is the total kinetic energy of the system. The term $\sum_i \vec{r}_i \cdot \vec{F}_i$ is called the virial V. Considering N3L,

$$V = \sum_i \sum_{j>i} \vec{F}_{j\to i} \cdot (\vec{r}_i - \vec{r}_j) \tag{2.4}$$

Note that for an ideal gas interacting through gravity alone:

$$V = -\sum_i \sum_{j>i} \frac{Gm_im_j}{r_{ij}} \tag{2.5}$$

which we realize is simply the potential energy of the system. Then:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2T + \Omega \tag{2.6}$$

If the system is in steady state, the time derivative vanishes, hence we have: $2T = -\Omega$ for a gravitating ideal gas in equilibrium.

2.1.3 Effect of external pressure

The Jean's equation is:

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Omega + \Sigma \tag{2.7}$$

where Σ is called the surface tension term, which arises if the system is embedded in an environment with a nonzero pressure.

Example Consider a uniform gas cloud with $\gamma = 5/3$, temperature T, radius r_{cl}, total mass M and gas mass M_{gas}. The total gravitational effect is hence contributed by the total mass, but the baryonic matter is only given by M_{gas}. Assume no bulk motions. Let there be an external pressure around the cloud p_{ext}. The specific thermal energy is given by:

$$u = \frac{k_B T}{(\gamma - 1)\mu m_H} \tag{2.8}$$

where μ is the mean molecular weight. The total kinetic energy is hence $T = M_{gas}u$. The total potential energy due to gravitational binding will only be calculated for the interaction between a thin spherical gas shell and the total enclosed mass. Hence we need to take ρ_{gas} and ρ_{tot} in the calculation, and obtain:

$$\Omega = -\frac{3G}{5}\frac{M_{gas}M}{r_{cl}} \tag{2.9}$$

The surface tension term is:

$$\Omega = -4\pi r_{cl}^3 P_{ext} \tag{2.10}$$

which is the work done by the external pressure. Putting these terms together,

$$2M_{gas}\frac{k_B T}{(\gamma - 1)\mu m_H} - \frac{3G}{5}\frac{M_{gas}M}{r_{cl}} - 4\pi r_{cl}^3 P_{ext} = 0 \tag{2.11}$$

The virial temperature can be obtained by solving for T in the absence of an external pressure (taking $\gamma = 5/3$):

$$T_{vir} = \frac{\mu m_H G M}{5 k_B r_{cl}} \tag{2.12}$$

Hence the heavier the cloud for the same radius, the hotter the cloud has to be to avoid collapse and remain at steady state. If there exists an external pressure, then the temperature of the cloud has to be higher than the virial temperature to prevent collapse.

2.2 Monday 25 Jan 2016

2.2.1 Equation of state for ideal gas

$$p = \frac{\mathcal{R}_*}{\mu}\rho T \tag{2.13}$$

where $\mathcal{R}_*$ is the modified gas constant numerically equal to $1000R = 8300JK^{-1}mol^{-1}$ and μ is the mean molecular weight in kilograms.

2.2.2 Barotropic equation of state

is such that pressure is only a function of density (no need to take temperature into account). For the adiabatic case:

$$p = K\rho^\gamma, \quad \gamma = \frac{C_p}{C_v} \tag{2.14}$$

Start from the 1st law (use total differentials for a reversible process):

$$dQ = d\epsilon + pdV \tag{2.15}$$

$$C_v \equiv \frac{d\epsilon(T)}{dT} \tag{2.16}$$

$$\implies dQ = C_v dT + \frac{\mathcal{R}_* T}{\mu V} dV, \quad \rho = \frac{1}{V} \tag{2.17}$$

Note that for a reversible process, entropy is a constant and $dQ = 0$. Dividing by T:

$$C_v d\ln T + \frac{\mathcal{R}_*}{\mu} d\ln V = 0 \tag{2.18}$$

$$\implies V \propto T^{-C_v\mu/\mathcal{R}_*} \tag{2.19}$$

$$\implies p \propto T^{1+C_v/(\mathcal{R}_*/\mu)} \tag{2.20}$$

Note that in the classical limit we can also write:

$$C_v = f\frac{\mathcal{R}_*}{2\mu} \tag{2.21}$$

where f is the number of degrees of freedom in the case. For a monoatomic gas, we have $C_v = \frac{3\mathcal{R}_*}{2\mu}$ and for the diatomic gas, we have $C_v = \frac{5\mathcal{R}_*}{2\mu}$.

We may now manipulate the 1st law by using the total derivative:

$$d(pV) = \frac{\mathcal{R}_*}{\mu} dT \tag{2.22}$$

$$dQ = \left(\frac{d\epsilon}{dT} + \frac{\mathcal{R}_*}{\mu}\right) dT - V dp \tag{2.23}$$

where the term in the parenthesis is the specific heat capacity at constant pressure. In terms of $\gamma \equiv \frac{C_p}{C_v}$, we hence have the scaling relations:

$$p \propto T^{\gamma/(\gamma-1)} \tag{2.24}$$

$$V \propto T^{-1/(\gamma-1)} \tag{2.25}$$

$$\implies p \propto V^{-\gamma} \tag{2.26}$$

$$\implies p = K\rho^\gamma \tag{2.27}$$

Note that K can either be constant locally (each fluid element has its own K and constant energy content) or globally (isentropic fluid where K is the same for every element).

2.2.3 Energy Equation

We now choose to be exceedingly confusing and define the work to be done by the fluid element to write:

$$\frac{D\epsilon}{Dt} = \frac{DW}{Dt} + \frac{DQ}{Dt} \tag{2.28}$$

The differential work done by the fluid element is defined $dW = -pdV \implies \frac{DW}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt}$ where we used $\rho = \frac{1}{V}$. We also define $\frac{DQ}{Dt} = -\dot{Q}_{cool}$, the cooling function. Then:

$$\frac{D\epsilon}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} - \dot{Q}_{cool} \tag{2.29}$$

Now we consider the total energy per unit volume in order to use the continuity equations:

$$E = \rho(\frac{1}{2}u^2 + \psi + \epsilon) \tag{2.30}$$

Using the definition of the Lagrangian total derivative:

$$\frac{DE}{Dt} = \left(\frac{\partial E}{\partial t} + \vec{u}\cdot\nabla E\right) = \frac{D\rho}{Dt}\frac{E}{\rho} + \rho\left(\vec{u}\cdot\frac{D\vec{u}}{Dt} + \frac{D\psi}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} - \dot{Q}_{cool}\right) \tag{2.31}$$

We can use the continuity equation to replace $\frac{D\rho}{Dt}$ and the momentum equation to replace $\frac{D\vec{u}}{Dt}$ and obtain:

$$\frac{\partial E}{\partial t} + \nabla\cdot[(E+p)\vec{u}] = \rho\frac{\partial\psi}{\partial t} - \rho\dot{Q}_{cool} \tag{2.32}$$

where we note that E has the same units of p because we are looking at the energy per unit volume.

2.3 Wednesday, 27 Jan 2016

2.3.1 Hydrostatic equilibrium

We set $\vec{u} = 0$ and set all time derivatives to zero. Then the continuity equation is trivially satisfied. The momentum equation in Eulerian form gives:

$$0 = -\nabla p - \rho\nabla\Psi \tag{2.33}$$

which is the condition for hydrostatic equilibrium. The other equation to use is Poisson's equation.

Now assume Barotropic flow $p(\rho)$.

2.3.2 Case 1: Isothermal slab

Use the ideal gas equation $p = \frac{\mathcal{R}_*\rho T}{\mu} \equiv A\rho$. Then the hydrostatic equilibrium condition gives:

$$A\frac{\nabla\rho}{\rho} = -\nabla\Psi \tag{2.34}$$

$$\implies A\frac{d\ln\rho}{dz} = -\frac{d\Psi}{dz} \implies \Psi = -A\ln\frac{\rho}{\rho_0} + \Psi_0 \tag{2.35}$$

$$\implies \rho = \rho_0 e^{-(\Psi-\Psi_0)/A} \tag{2.36}$$

Poisson's equation requires:

$$\frac{d^2\Psi}{dz^2} = 4\pi G\rho_0 e^{-(\Psi-\Psi_0)/A} \tag{2.37}$$

Make the change of variables:

$$\chi = -\frac{\Psi-\Psi_0}{A} \tag{2.38}$$

$$Z = z\sqrt{\frac{2\pi G\rho}{A}} \tag{2.39}$$

which gives the new DE:

$$\frac{d^2\chi}{dZ^2} = -2e^{\chi} \tag{2.40}$$

with boundary conditions:

$$\chi(Z=0) = 0 \tag{2.41}$$

$$\left.\frac{d\chi}{dZ}\right|_{Z=0} = 0 \tag{2.42}$$

Rearranging the DE:

$$\frac{1}{2}\frac{d}{dZ}\left[\left(\frac{d\chi}{dZ}\right)^2\right] = -2\frac{d(e^{\chi})}{dZ} \tag{2.43}$$

$$\implies \left(\frac{d\chi}{dZ}\right)^2 = C_1 - 4e^{\chi}, \quad C_1 = 4 \tag{2.44}$$

$$\implies \frac{d\chi}{dZ} = 2\sqrt{1-e^{\chi}} \tag{2.45}$$

$$\implies -2\tanh^{-1}\sqrt{1-e^{\chi}} + C_2 = 2Z, \quad C_2 = 0 \tag{2.46}$$

$$\implies \chi = \ln(1-\tanh^2(-Z)) \tag{2.47}$$

$$\implies \chi = \ln(1-\tanh^2 Z) \tag{2.48}$$

$$\rho = \frac{\rho_0}{\cosh^2\left(z\sqrt{\frac{2\pi G\rho_0}{A}}\right)} \tag{2.49}$$

2.3.3 Case 2: Earth's Atmosphere

We want to solve the ODE:

$$\frac{\nabla p}{\rho} = -\nabla\Psi = -g\hat{r} \tag{2.50}$$

which is an exponential atmosphere:

$$\rho = \rho_0 e^{\frac{-\mu g z}{\mathcal{R}_* T}} \tag{2.51}$$

2.3.4 Stars

Proceed in spherical polar coordinates. The hydrostatic equilibrium condition is:

$$\frac{dp}{dr} = -\rho\frac{d\Psi}{dr} \tag{2.52}$$

We require that the density be positive and we note that since all the parameters are radially symmetric, we can just write the pressure and density as functions of the gravitational potential:

$$p(\Psi), \quad \rho(\Psi) \tag{2.53}$$

Hence the surfaces of constant density, pressure and gravitational potential coincide, and we call this situation a barotrope.

2.3.5 Polytropes - another barotropic equation of state

Let:

$$p = K\rho^{1+1/n} \tag{2.54}$$

where n is the barotropic index. Then the density can be written as a function of the gravitational potential:

$$\rho = \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n \tag{2.55}$$

where ρ_c is the density at the center of the star. The boundary conditions are:

$$\rho = 0 \quad \text{at surface} \tag{2.56}$$

$$\Psi_c, \rho_c \quad \text{at r=0} \tag{2.57}$$

Non-dimensionalizing the equation:

$$\theta = \frac{\Psi_T - \Psi}{\Psi_T - \Psi_c} \tag{2.58}$$

$$\implies \nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n \tag{2.59}$$

Introducing the scaled radius:

$$\xi = \sqrt{\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}}r \tag{2.60}$$

Then the relation between the radius and the gravitational potential becomes:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n}$$

which is called the Lane-Emden equation.

Chapter 3

Week 3

3.1 Friday 29 Jan 2015

3.1.1 Solution to Lane-Emden Equations

Recall that we wanted to solve:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n}$$

Consider $n = \infty$ so that the pressure is proportional to the density, which is the isothermal condition. The solution will be that $\rho \propto r^{-2} \implies M \propto r$. Objects that satisfy this are called Bonner-Ebert spheres. Since the mass included goes to infinity as r goes to infinity, we need to truncate the star at a physical radius, then treat the effect of the external mass as exerting an external pressure stabilizing the star.

3.1.2 Scaling relations

For a fixed polytropic index n, assume that the proportionality constant K from $P = K\rho^{1+1/n}$ is the same for each star in the family of solutions. Then we want to see how each parameter scales with the central density ρ_c. Eliminating $\Psi_T - \Psi_c$ from the dimensionless parameters, we obtain that:

$$M \propto \rho_c^{(3/n-1)/2} \tag{3.1}$$

$$r \propto \rho_c^{(1/n-1)/2} \tag{3.2}$$

$$\implies M \propto r^{(3-n)/(1-n)} \tag{3.3}$$

If the star has an adiabatic equation of state with $\gamma = 5/3 \implies n = \frac{3}{2} \implies M \propto r^{-3}$. This, however, is not observed. We observe that $M \propto r$ instead. It is not true that K is constant for a fixed value of n. A better approximation is to take the core temperature T_c to be the same across the family. Then at the center, we have a relation between K and ρ_c:

$$K \propto \rho_c^{-1/n} \tag{3.4}$$

and this will give $M \propto r$.

3.1.3 Timescales of hydrostatic equilibrium

The propagation of disturbances through the fluid is given by the speed of sound:

$$t_h \sim \frac{r}{c_s} \tag{3.5}$$

This is approximately 1 day for the sun $t_{\odot,h}$.

3.1.4 Thermal timescale

$$t_{th} \sim \frac{E_{pot}}{L} \tag{3.6}$$

where L is the luminosity. For the Sun, $t_{\odot,th} = 30$Myrs. Hence the hydrostatic timescale is much faster than the thermal timescale.

The relative size of these timescales gives the mode by which the star adapts its radius to an amount of mass dumped on the surface. Note that for an adiabatic change, mass scales as $M \propto r^{(3-n)/(1-n)}$. Hence for an increase in mass, the star will move towards the adiabatic scaling line in the short term (short hydrostatic equilibrium timescale). In the long term (thermal timescale), the star will move back to the scaling line for $M \propto r$ for the constant central temperature.

3.1.5 Accreting rotating star

Consider a star with angular velocity Ω. Let a small amount of mass ΔM be added. We conserve angular momentum:

$$J \propto M r^2 \Omega \tag{3.7}$$

Hence for a change $\Omega \rightarrow \Omega + \Delta\Omega$, we require that:

$$\frac{\Delta\Omega}{\Omega} = -\frac{\Delta(Mr^2)}{Mr^2} \tag{3.8}$$

Now we model the addition of mass as an adiabatic process. Hence we have a relation between the mass and radius, and:

$$\frac{\Delta\Omega}{\Omega} = -\left(\frac{5-3n}{3-n}\right)\frac{\Delta M}{M} \tag{3.9}$$

Hence the sign of the angular velocity change will be determined by the sign of the term in the parenthesis.

3.2 Monday 1 Feb 2016

3.2.1 Case 1: Uniform medium

Consider the continuity equation and the momentum equation in the absence of gravity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{3.10}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P \tag{3.11}$$

Let the unperturbed system have uniform pressure P_0, density ρ_0 and zero velocity $\vec{u}_0 = \vec{0}$. Let it be at steady state. Now consider a perturbation of the form:

$$P = P_0 + \Delta P \tag{3.12}$$

$$\rho = \rho_0 + \Delta\rho \tag{3.13}$$

$$\vec{u} = \Delta\vec{u} \tag{3.14}$$

These are Lagrangian perturbations, that is, perturbations to a single fluid element.

3.2.2 Eulerian perturbations

Interconvert using the Lagrangian derivative:

$$\delta\rho = \Delta\rho - \vec{\xi}\cdot\nabla\rho \tag{3.15}$$

where δ represents Eulerian perturbations and Δ represents Lagrangian perturbations.

Now $\vec{\xi}$ is the differential displacement of the fluid element.

Note that in this uniform case, the gradient of the unperturbed quantities vanishes, hence the Lagrangian perturbation is exactly equal to the Eulerian perturbation. Substituting these perturbations into the fluid equations and doing grungy algebra to first order, we end up with:

$$\frac{\partial\Delta\rho}{\partial t} + \rho_0\nabla\cdot\Delta\vec{u} = 0 \tag{3.16}$$

$$\frac{\partial\Delta\vec{u}}{\partial t} = -\frac{1}{\rho_0}\nabla(\Delta P) \tag{3.17}$$

Now we implement an additional assumption that the fluid is barotropic $P(\rho) \implies \nabla P(\rho) = \frac{dP}{d\rho}\nabla\rho$. Hence:

$$\frac{\partial\Delta\vec{u}}{\partial t} = -\frac{1}{\rho_0}\frac{dP}{d\rho}\nabla(\Delta\rho) \tag{3.18}$$

Combining,

$$\frac{\partial^2\Delta\rho}{\partial t^2} = \frac{dP}{d\rho}\nabla^2(\Delta\rho) \tag{3.19}$$

which is a wave equation. Moving into the Fourier domain, let $\Delta\rho = \Delta\rho_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$-\omega^2 = -\frac{dP}{d\rho}k^2 \tag{3.20}$$

$$\implies v_p = \frac{\omega}{k} = \sqrt{\frac{dP}{d\rho}} \equiv c_s \tag{3.21}$$

Substituting this ansatz into the $\Delta\vec{u}$ solution, we obtain that the velocity perturbation is given by:

$$\Delta\vec{u} = \frac{\omega}{k}\frac{\Delta\rho_0}{\rho_0}e^{i(\vec{k}\cdot\vec{x}-\omega t)}\hat{k} \tag{3.22}$$

Hence the velocity perturbation is in phase with the density perturbation. Also, since the density perturbation is small, we have:

$$|\Delta\vec{u}| << c_s \tag{3.23}$$

Observe that the functional equation $P(\rho)$ depends on whether the sound propagation is conducted in an isothermal manner or adiabatic manner. This may or may not be the same condition as the unperturbed medium. This means that a medium that is initially isothermal can experience adiabatic perturbations. In each case:

$$c_{s,T} = \sqrt{\frac{\mathcal{R}_* T}{\mu}} \tag{3.24}$$

$$c_{s,A} = \sqrt{\frac{\mathcal{R}_* \gamma T}{\mu}} \tag{3.25}$$

3.3 Wednesday 03 Feb 2016

3.3.1 Sound propagation in non-uniform medium

Consider sound waves propagating in an isothermal atmosphere experiencing constant gravitational acceleration. By symmetry, sound waves moving in the horizontal direction are not affected by the atmosphere. We hence examine sound waves in the vertical direction and consider all functions of z alone. Then by the continuity and momentum equations (u is velocity in the z-direction):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \tag{3.26}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \tag{3.27}$$

The equilibrium solution corresponds to $u_0 = 0$ and $\rho_0(z) = \rho_0 e^{-z/H}, H = \frac{\mathcal{R}_* T}{\mu g}$. The pressure is also $P_0(z) = P_0 e^{-z/H}$ as well. We hence consider a Lagrangian perturbation about this equilibrium state:

$$u = \Delta u \tag{3.28}$$

$$\rho = \rho_0(z) + \Delta\rho \tag{3.29}$$

$$P = P_0(z) + \Delta P \tag{3.30}$$

Recall that the relation between Lagrangian and Eulerian perturbations was:

$$\delta\rho = \Delta\rho - \vec{\xi}\cdot\nabla\rho \tag{3.31}$$

Then the Eulerian perturbations are:

$$\delta\vec{u} = \Delta\vec{u} \tag{3.32}$$

$$\delta\rho = \Delta\rho - \xi_z\frac{\partial\rho_0(z)}{\partial z} \tag{3.33}$$

$$\delta P = \Delta P - \xi_z\frac{\partial P_0(z)}{\partial z} \tag{3.34}$$

Note that the differential displacements are related to the velocity by:

$$\Delta\vec{u} = \frac{d\vec{\xi}}{dt} = \frac{\partial\vec{\xi}}{\partial t} + \vec{u}\cdot\nabla\vec{\xi} = \frac{\partial\vec{\xi}}{\partial t}, \quad \vec{u}_0 = 0 \tag{3.35}$$

Substituting these Eulerian perturbations into the Eulerian continuity and momentum equations, and sifting through the grungy algebra, we obtain:

$$\frac{\partial\Delta\rho}{\partial t} + \rho_0\frac{\partial\Delta u_z}{\partial z} = 0 \tag{3.36}$$

$$\frac{\partial\Delta u_z}{\partial t} = -\frac{c_u^2}{\rho_0}\frac{\partial\Delta\rho}{\partial z}, \quad c_u = \sqrt{\frac{dP_0(z)}{d\rho_0(z)}} \tag{3.37}$$

Combining,

$$\frac{\partial^2\Delta\rho}{\partial t^2} = \rho_0(z)\frac{\partial}{\partial z}\left(\frac{c_u^2}{\rho_0(z)}\frac{\partial\Delta\rho}{\partial z}\right) \tag{3.38}$$

Note that $\frac{dP_0(z)}{d\rho_0(z)}$ = constant. Then we have:

$$\frac{\partial^2\Delta\rho}{\partial t^2} - c_u^2\frac{\partial^2\Delta\rho}{\partial z^2} - \frac{c_u^2}{H}\frac{\partial\Delta\rho}{\partial z} = 0 \tag{3.39}$$

Moving into the Fourier domain, we make the ansatz $\Delta\rho \propto e^{i(kz-\omega t)}$. Then we obtain the dispersion relation:

$$-\omega^2 + c_u^2 k^2 - \frac{c_u^2}{H} ik = 0 \implies \omega^2 = c_u^2 \left(k^2 - \frac{ik}{H}\right) \tag{3.40}$$

Inverting,

$$k(\omega) = \frac{i}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}} \tag{3.41}$$

Note that the sign of the quantity in the square root determines if $k(\omega)$ has any real propagating part. If $\omega > \frac{c_u}{2H}$, then the square root gives a real part and hence the sound wave propagates. Then the density goes as:

$$\Delta\rho(z) = Ae^{-z/2H} \exp\left[i\left(\sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}} z - \omega t\right)\right] \tag{3.42}$$

The velocity amplitude can also be determined:

$$\Delta u = \frac{\Delta\rho}{\rho_0}\frac{\omega}{k} \implies \Delta u \propto e^{z/2H} \tag{3.43}$$

Note that this implies that the velocity perturbation amplitude will increase if it moves in the positive z direction. This is not physically possible generally since we have ignored viscosity and considered an isothermal atmosphere.

On the other hand, if $\omega < \frac{c_u}{2H}$, then the values of k are purely imaginary. Proceeding with the same analysis as above, we obtain:

$$\Delta u = Ae^{kz}e^{i\omega t}, \Delta\rho = Be^{kz}e^{i\omega t} \tag{3.44}$$

Hence there is no propagation, but the amplitude still depends on the height z. We just have standing waves.

3.3.2 Transmission of sound waves across boundary

Consider non-dispersive propagation of sound waves. Then to the left of the boundary, we have:

$$\Delta\rho = e^{i(k_1 x - \omega_1 t)} + re^{i(k_3 x - \omega_3 t)} \tag{3.45}$$

and to the right, we have:

$$\Delta\rho = te^{i(k_2 x - \omega_2 t)} \tag{3.46}$$

Let the sound velocities be $c_{s,1}$ and $c_{s,2}$ for the left and right media respectively.

Now we require that the wave be continuous across the boundary. This requires that $\omega_1 = \omega_2 = \omega_3$ and $k_1 = -k_3$. We also require that the amplitude be continuous across the boundary, which gives:

$$1 + r = t \tag{3.47}$$

Requiring that the first derivative of the wave also be continuous, we obtain that:

$$k_1(1-r) = k_2(t) \tag{3.48}$$

Solving for r and t simultaneously, we obtain:

$$r = \frac{k_1 - k_2}{k_1 + k_2} \tag{3.49}$$

$$t = \frac{2k_1}{k_1 + k_2} \tag{3.50}$$

Note that if the second medium is extremely cold, so that $c_{s,2}$ is small and k_2 is large, then the transmitted wave amplitude goes to zero.

Chapter 4

Week 4

4.1 Friday, 5 Feb 2015

4.1.1 Supersonic flows

In the subsonic regime, the vector addition of the subsonic fluid flow and the disturbance (assumed to be moving spherically outward at the speed of sound) can point in all directions (covers 4π solid angle). This communicates information of the disturbance to all regions of the fluid. However, in supersonic flow, the vector sum of the supersonic velocity and the disturbance displacement vector will always have a component in the direction of fluid flow. Hence there is no information about the flow proceeding in the opposite direction to the supersonic flow. The information propagates along a Mach cone.

The half-angle of the cone α is given by:

$$\sin\alpha = \frac{c_s}{v} \equiv \frac{1}{\mathcal{M}} < 1 \tag{4.1}$$

4.1.2 Planar Shock

Let the incoming flow be characterized by u_1, ρ_1, P_1 and the outgoing flow be characterized by u_2, ρ_2, P_2. Put the shock along the plane $x = 0$. Continuity gives:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \tag{4.2}$$

Integrate the continuity equation around the shock using a box with infinitesimal width. Then we have (exchanging the order of derivatives and using the fundamental theorem of calculus):

$$\frac{\partial}{\partial t}\int \rho dx + \rho u_x|_{x=dx/2} - \rho u_x|_{x=-dx/2} = 0 \tag{4.3}$$

Now we require that the mass flux be conserved across the shock (no build-up of mass in the shock), so that the first term vanishes. This gives the first Rankine-Hugoniot condition:

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

where all velocities u are assumed to be in the x direction. Generally the velocity is lower in the post shock region $u_2 < u_1$ so the density in the post shock region is higher than in the pre-shock region.

Now consider the momentum equation for the gravitational potential and the ram pressure:

$$\frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial}{\partial x}(\rho u_x u_x + P) - \rho \frac{d\Psi}{dx} \tag{4.4}$$

Suppose that the gravitational potential is constant in space and in time. Then we throw away the last term. Integrate the DE across the shock, and note that the mass flux across the shock does not vary, allowing us to let the first term vanish as well:

$$\boxed{\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2}$$

This second R-H condition says that the sum of the ram pressure and thermal pressure is continuous across the shock. We may interpret the shock as redistribution ordered ram pressure motions into disordered thermal motions.

Now assume that the fluid flows adiabatically. That is, we do not allow fluid elements to exchange energy. We also disallow cooling: $\dot{Q}_{cool} = 0$. Then the energy equation gives:

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\vec{u}] = -\rho \dot{Q}_{cool} + \rho \frac{\partial \Psi}{\partial t} \tag{4.5}$$

We ignore the terms in the RHS. Integrating the resultant DE across the shock, we obtain:

$$\frac{\partial}{\partial t}\int E dx + (E + P)u_x|_{x=dx/2} - (E + P)u_x|_{x=-dx/2} = 0 \tag{4.6}$$

We require that the total energy of the fluid element be conserved across the shock. Then the energy flux across the shock has to be conserved, and we note that the first term goes to zero. This gives us the third R-H equation:

$$\boxed{(E_1 + P_1)u_1 = (E_2 + P_2)u_2}$$

Note further that the total energy of the fluid element can be written as:

$$E = \rho \left(\frac{1}{2}u^2 + \epsilon + \Psi \right) \tag{4.7}$$

where ϵ is the internal energy density. We substitute this expression into the third equation and divide both sides by $\rho_1 u_1 = \rho_2 u_2$ (1st R-H condition) to obtain an alternative formulation of the third R-H condition:

$$\boxed{\frac{1}{2}u_1^2 + \epsilon_1 + \frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \epsilon_2 + \frac{P_2}{\rho_2}}$$

Call the sum $\epsilon + \frac{P}{\rho}$ the enthalpy. Hence the third R-H condition tells use that the sum of the energy and the enthalpy terms are continuous across the shock. Observe that the KE for the pre-shock region is higher than the KE for the post-shock region. Hence the enthalpy of the post-shock region is higher than that of the pre-shock region. The shock converts KE into enthalpy.

We may further re-write the internal energy per unit mass of the flow to be:

$$\epsilon = c_v T = \frac{c_v}{\mathcal{R}_*/\mu}\frac{P}{\rho} = (\gamma - 1)\frac{P}{\rho} \tag{4.8}$$

which follows from:

$$\gamma = \frac{c_p}{c_v}, \quad c_p - c_v = \frac{\mathcal{R}_*}{\mu} \implies c_v(\gamma - 1) = \frac{\mathcal{R}_*}{\mu} \tag{4.9}$$

Now assume that the adiabatic factor γ does not change across the shock. Then we have the simplified 3rd R-H condition using the adiabatic sound velocity $c_s = \sqrt{\frac{\gamma P}{\rho}}$:

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2} \tag{4.10}$$

$$\implies \frac{1}{2}u_1^2 + \frac{c_{s1}^2}{\gamma - 1} = \frac{1}{2}u_2^2 + \frac{c_{s2}^2}{\gamma - 1} \tag{4.11}$$

Some grungy algebra later...

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)P_2 + (\gamma - 1)P_1}{(\gamma + 1)P_1 + (\gamma - 1)P_2} = \frac{u_1}{u_2} \tag{4.12}$$

Strong shock limit Let $P_2 >> P_1$. Then $\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma+1}{\gamma-1}$ which is approximately 4 for a $\gamma = 5/3$. Hence for an adiabatic shock, the limiting ratio of the densities is 4.

Consider a fluid element crossing the shock. Note that even though P_1/P_2 can achieve a wide range, ρ_1/ρ_2 is bound. This means that $K = P/\rho^\gamma$ is not conserved across the shock. There is an entropy jump across the shock - ordered bulk flow is converted into disordered thermal flow.

4.2 Monday, 8 Feb 2015

4.2.1 Isothermal shocks

Note that as compared to the adiabatic shock, we may now take $\dot{Q}_{cool} \neq 0$. We also require $T_1 = T_2$. Note that this does not imply that the shock region itself has the same temperature. The temperature of the fluid element increases as it enters the shock, and then as it exits the shock, it cools back to a constant temperature (not necessarily the same temperature) over a lengthscale called the cooling length l_{cool}. If the cooling length is on the order of the size of the system, then the system behaves effectively adiabatically. If we take the cooling length to be small compared to the size of the system and assume that the asymptotic temperature is the same across the shock, then we obtain an isothermal shock approximation.

In the isothermal case, the first two R-H conditions still hold:

$$\rho_1 u_1 = \rho_2 u_2 \tag{4.13}$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \tag{4.14}$$

Since the temperature is the same on both sides of the shock, we take $c_{s1} = c_{s2} = c_s = \sqrt{\frac{\mathcal{R}_* T}{\mu}}$ and $P_{1,2} = c_s^2 \rho_{1,2}$. Combining the two R-H conditions, we have:

$$c_s^2 = u_1 u_2 \tag{4.15}$$

This gives the density relation:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_s}\right)^2 = M_1^2 > 1 \tag{4.16}$$

Note further that for a shock, we require that $u_1 > c_s$. Hence the flow after the shock must be subsonic $u_2 < c_s$. This condition (supersonic u_1 implies subsonic u_2) holds also for adiabatic shocks, but is algebraically messy.

Spherically symmetric example Let there be a spherical cloud with ρ_1, P_1, T_1 that is stationary $u_1 = 0$ embedded in a medium with ρ_2, P_2, T_2 that is also stationary $u_2 = 0$. Assume that $T_1 < T_2$ so the cloud is colder than its environment, and let it be denser $\rho_1 > \rho_2$. Neglect gravity. If the pressures are equal, then the only two processes that occur are diffusion and conduction.

If, however, $u_2 > c_s$, then a shock will form. The shock will be curved due to the spherical symmetry of the cloud, and is called a bow shock.

4.3 Wednesday 10 Feb 2016

4.3.1 Blast waves

Spherically symmetric expanding shock. At $t = 0$, let the medium be uniform with density ρ_0. Consider a point explosion with some associated energy E. Assume that the temperature of the surrounding medium is approximately zero so that the sound speed is negligible. The blast wave Mach number hence goes to infinity. Also neglect the other pressures in the medium. Let the radius of the blast wave be R and let its thickness (small compared to R) be D. Let the blast wave shell have density and pressure ρ_1, P_1. Let the pressure in the cavity be P_{in}.

Since this blast wave is an extremely strong shock, in the adiabatic regime, the ratio of the densities approaches the limiting value of $\frac{\rho_1}{\rho_0} = 4$ for $\gamma = 5/3$.

Let the blast wave sweep most of the mass into its shell. Then we have:

$$\frac{4\pi}{3}\rho_0 R^3 = 4\pi\rho_1 R^2 D \tag{4.17}$$

$$\implies D = \frac{1}{3}\frac{\gamma - 1}{\gamma + 1}R \tag{4.18}$$

Now move into the frame of the shock. Then we may implement continuity:

$$\rho_0 u_0 = \rho_1 u_1 \tag{4.19}$$

and the relative velocity of the shock is:

$$U = u_0 - u_1 \approx \frac{2u_0}{\gamma + 1} \tag{4.20}$$

in the limit of $\rho_1/\rho_0 = (\gamma + 1)/(\gamma - 1)$.

Now make the further assumption that the pressure in the interior of the blast radius is linearly related to the shock pressure:

$$P_{in} = \alpha P_1 \tag{4.21}$$

Implementing the second R-H condition:

$$P_0 + \rho_0 u_0^2 = P_1 + \rho_1 u_1^2 \tag{4.22}$$

$$\implies P_1 = \rho_0 u_0^2 \left(1 - \frac{\rho_1 u_1^2}{\rho_0 u_0^2}\right), \quad P_0 \approx 0 \tag{4.23}$$

$$\implies P_1 = \frac{2}{\gamma+1}\rho_0 u_0^2 \tag{4.24}$$

Now the pressure in the interior of the blast wave will exert a force on the blast wave, and this should result in a time rate of change in blast wave momentum. Hence:

$$\frac{d}{dt}\left(\frac{4\pi}{3}\rho_0 R^3 U\right) = \frac{d}{dt}\left(\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma+1}\right) = 4\pi R^2 P_{in} = 4\pi R^2 \alpha \frac{2}{\gamma+1}\rho_0 u_0^2 \tag{4.25}$$

$$\implies \frac{d(R^3 u_0)}{dt} = 3\alpha R^2 u_0^2 \tag{4.26}$$

Note further that u_0 is the velocity of the shock in the frame of the unperturbed medium, which means that we can write $u_0 = \frac{dR}{dt}$. Assume a power law solution. Then the differential equation and ansatz are:

$$\frac{d}{dt}(R^3 \dot{R}) = 3\alpha R^2 \dot{R}^2 \tag{4.27}$$

$$\text{Let } R(t) \propto t^b \tag{4.28}$$

$$\implies b^2(4-3\alpha) = b \tag{4.29}$$

$$\implies b = \frac{1}{4-3\alpha}, \quad b \neq 0 \tag{4.30}$$

$$\implies R \propto t^{1/(4-3\alpha)}, \quad u_0 \propto t^{(3\alpha-3)/(4-3\alpha)} \tag{4.31}$$

Note that the kinetic energy of the blast wave is:

$$KE = \frac{1}{2}\left(\frac{4\pi}{3}\rho_0 R^3\right) U^2 \tag{4.32}$$

while the internal energy is largely contained in the cavity:

$$\epsilon = \frac{1}{\gamma-1} P_{in} \tag{4.33}$$

where ϵ is the internal energy per unit mass. Hence the total internal energy is:

$$\epsilon V = \frac{4\pi}{3} R^3 \frac{\alpha P_1}{\gamma-1} \tag{4.34}$$

By conservation of energy, we hence have:

$$E = \frac{1}{2}\left(\frac{4\pi}{3}\rho_0 R^3\right) U^2 + \frac{4\pi}{3} R^3 \frac{\alpha P_1}{\gamma-1} \tag{4.35}$$

$$\implies E \propto R^3 u_0^2 \propto t^{(6\alpha-3)(4-3\alpha)} \tag{4.36}$$

Since energy is conserved, we require $\alpha = 1/2$.

Chapter 5

Week 5

5.1 Friday, 12 Feb 2016

5.1.1 Blast Waves - Similarity Solution

Recall that the relevant parameters are the explosion energy E and unperturbed density of the surrounding medium ρ_0. If these are the only parameters of interest, it is not possible to obtain an expression with units of length. Hence there is no natural length scale in the problem. Introducing a time parameter, we can, however, obtain the length parameter than depends on time:

$$\lambda = \left(\frac{Et^2}{\rho_0}\right)^{1/5} \tag{5.1}$$

The dimensionless distance can hence be written as $\xi \equiv \frac{r}{\lambda}$.

We also examine solutions for $X \in \{\rho, P, u, T, v\}$ in separated form: $X(r,t) = X_1(t)\tilde{X}(\xi(r,t))$. Also assume spherical symmetry. Then by the chain rule:

$$\left.\frac{\partial X}{\partial r}\right|_t = X_1 \frac{d\tilde{X}}{d\xi} \left.\frac{d\xi}{dr}\right|_t \tag{5.2}$$

$$\left.\frac{\partial X}{\partial t}\right|_t = \tilde{X}\frac{dX_1}{dt} + X_1 \frac{d\tilde{X}}{d\xi} \left.\frac{d\xi}{dt}\right|_r \tag{5.3}$$

$$\tag{5.4}$$

5.1.2 Breakdown of similarity solution

Note that by taking the temperature of the surrounding medium to be near zero, we have neglected the pressure due to the interstellar medium. This does not hold in general. This approximation should breakdown when the pressure in the shell becomes on the order of the ambient pressure. Recall that the shell pressure is given by:

$$P_1 = \frac{2}{\gamma+1}\rho_0 u_0^2 \tag{5.5}$$

hence the approximation should breakdown around:

$$\frac{2}{\gamma+1}\rho_0 u_0^2 = P_1 = \frac{\rho_0 c_s^2}{\gamma} = P_0 \tag{5.6}$$

$$\implies u_0 \approx c_s \tag{5.7}$$

Hence when the shell is no longer moving supersonically, the approximation breaks down.

We may also estimate the energy of the system in this situation:

$$u_0^2 \sim \frac{\gamma+1}{2\gamma} c_s^2 \tag{5.8}$$

$$\implies E(P_0 \approx P_1) \sim \frac{4\pi}{3} \rho_0 R_{max}^3 \frac{c_s^2}{2\gamma} \frac{3\gamma - 1}{\gamma^2 - 1} \tag{5.9}$$

where we note that R_{max} marks the radius past which the blast wave does not propagate significantly (since the blast velocity becomes subsonic and sound waves are released outwards instead). Note further that the total internal energy can be written as:

$$\epsilon = V \frac{P_0}{\gamma - 1} = \frac{4\pi}{3} R_{max}^3 \frac{c_s^2 \rho_0/\gamma}{\gamma - 1} \tag{5.10}$$

Comparing this to the total energy, we note that at R_{max}, the total energy is on the order of the internal energy. Hence most of the energy of the explosion has been converted into the internal energy of the cavity region within the maximum radius of the blast wave.

5.1.3 Non-self similar solution with more parameters

We now have three parameters in the model: $E, \rho_0, T_0(c_s)$. We may now construct a natural length scale and a time scale:

$$r \sim \left(\frac{E}{\rho_0 c_s^2} \right)^{1/3} \tag{5.11}$$

$$t \sim \frac{1}{c_s} \left(\frac{E}{r_0 c_s^2} \right)^{1/3} \tag{5.12}$$

5.1.4 Example: ISM

Let the parameters of the ISM be $T \sim 10^4, \rho \sim 10^{-21} kgm^{-3}$. We also know that $R_{max} \sim 100pc$ and $t_{max} \sim 10Myrs$ and the supernova rate $10^{-7} Myr^{-1} pc^{-3}$. Comparing the volume of the typical supernova with the size of a region with 1 supernova per time scale ($10^6 pc$), we note that the supernova blast volume exceeds $10^6 pc$, which implies that the entire ISM should be heated to blast wave temperatures of around $10^6 K$. This is clearly not the case since we see star formation. This is an indication that our adiabatic assumption is breaking down. There is significant cooling and the interstellar region is not uniform. The blast wave will get stopped when it goes subsonic. Also, the blast wave does not sweep up all the material — the cold neutral matter in the ISM are sufficiently cold and dense so that they do not get completely ablated by the oncoming blast wave.

5.2 Monday, 15 Feb 2016

5.2.1 Bernoulli's equation

Begin with the momentum equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla P - \nabla \Psi \tag{5.13}$$

Assume barotropic equation of state $P = P(\rho)$ and steady flow so that the time derivative vanishes. Recall that the material derivative can be written as:

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla(\frac{1}{2}u^2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) \tag{5.14}$$

Also define the vorticity $\boldsymbol{w} = \nabla \times \boldsymbol{u}$. For a barotropic flow, we also have:

$$\frac{1}{\rho}\nabla P = \nabla \int \frac{dP}{\rho} \tag{5.15}$$

Then the momentum equation becomes:

$$\nabla(\frac{1}{2}u^2) - \boldsymbol{u} \times \boldsymbol{w} = -\nabla \left[\int \frac{dP}{\rho} + \Psi \right] \tag{5.16}$$

Dotting both sides with $\boldsymbol{u}$, and noting that the LHS will vanish, we obtain:

$$\boldsymbol{u} \cdot \nabla \left[\frac{1}{2}u^2 + \int \frac{dP}{\rho} + \Psi \right] = 0 \tag{5.17}$$

Define $H = \frac{1}{2}u^2 + \int \frac{dP}{\rho} + \Psi$, the Bernoulli constant.

5.2.2 Steady, Irrotational flows

For $\boldsymbol{w} = 0$, we can write:

$$\nabla H = 0 \tag{5.18}$$

hence H is constant throughout.

5.2.3 Helmholtz Equation - General flow with vorticity

Write:

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla H + \boldsymbol{u} \times \boldsymbol{w} \tag{5.19}$$

$$\implies \frac{\partial \boldsymbol{w}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{w}) \tag{5.20}$$

Note that if the vorticity is initially zero, it will remain zero.

The flux of the vorticity over a surface S is constant and moves with the flow. This means that the Lagrangian derivative vanishes:

$$\frac{D}{Dt}\int_S \boldsymbol{w}\cdot d\boldsymbol{S} = \int_S \frac{\partial \boldsymbol{w}}{\partial t}\cdot d\boldsymbol{S} + \int_S \boldsymbol{w}\cdot \frac{Dd\boldsymbol{S}}{Dt} \tag{5.21}$$

$$= \int_S \frac{\partial \boldsymbol{w}}{\partial t}\cdot d\boldsymbol{S} + \int_S \boldsymbol{w}\cdot \oint_{d\boldsymbol{S}} \boldsymbol{u}\times d\boldsymbol{l} \tag{5.22}$$

$$= \int_S \frac{\partial \boldsymbol{w}}{\partial t}\cdot d\boldsymbol{S} + \int_S \oint_{d\boldsymbol{S}} \boldsymbol{w}\times\boldsymbol{u}\cdot d\boldsymbol{l} \tag{5.23}$$

$$= \int_S \frac{\partial \boldsymbol{w}}{\partial t}\cdot d\boldsymbol{S} + \oint_{C(S)} \boldsymbol{w}\times\boldsymbol{u}\cdot d\boldsymbol{l} \tag{5.24}$$

$$= \int_S \frac{\partial \boldsymbol{w}}{\partial t}\cdot d\boldsymbol{S} + \int_S \nabla\times(\boldsymbol{w}\times\boldsymbol{u})\cdot d\boldsymbol{S} \tag{5.25}$$

$$= 0 \tag{5.26}$$

5.2.4 Velocity potential

For an irrotational flow $\boldsymbol{w} = 0$ and incompressible flow $\nabla\cdot\boldsymbol{u} = 0$, we may define a potential function Φ_m such that $\boldsymbol{u} = -\nabla\Psi_m$ which satisfies the Poisson equation as well (using incompressibility condition): $\nabla^2\Psi_m = 0$.

5.3 Wednesday, 17 Feb 2016

5.3.1 De Laval nozzle

Steady flow through a pipe with variable cross section. Let the cross section be $A(z)$. Consider barotropic,irrotational flow. By continuity:

$$\dot{M} = \rho u A \tag{5.27}$$

$$\implies \ln\rho + \ln u + \ln A = \ln\dot{M} \tag{5.28}$$

$$\implies \nabla\ln\rho + \nabla\ln u + \nabla\ln A = \nabla\ln\dot{M} \tag{5.29}$$

$$\implies \frac{1}{\rho}\nabla\rho = -\nabla\ln u - \nabla\ln A, \quad \nabla\ln\dot{M} = 0 \tag{5.30}$$

Recall further that by the momentum equation for steady state flow without external forces:

$$\vec{u}\cdot\nabla u = -\frac{1}{\rho}\nabla P \tag{5.31}$$

and hence combining,

$$\vec{u}\cdot\nabla u = c_s^2\left(\nabla\ln u + \nabla\ln A\right) \tag{5.32}$$

Rearranging,

$$(u^2 - c_s^2)\nabla\ln u = c_s^2\nabla\ln A \tag{5.33}$$

Observe that when the pipe has a minimum or maximum cross section, then the RHS is vanishing and hence either $u = c_s$ (sonic transition) or an extremum in velocity.

Observe that if $u < c_s$ and the cross sectional area is decreasing, then u is increasing. However, if $u > c_s$ and the cross sectional area is *increasing*, u will also increase! This is due to the great variations of density in the supersonic regime, so the liquid can be compressed to a great extent.

5.3.2 Isothermal De Laval nozzle

Consider the Bernoulli equation without gravity:

$$\frac{1}{2}u^2 + \int \frac{dP}{\rho} = \text{constant} \tag{5.34}$$

The isothermal equation of state is:

$$P = \frac{\mathcal{R}_* \rho T}{\mu} \implies \int \frac{dP}{\rho} = c_s^2 \ln \rho \tag{5.35}$$

Suppose the sonic transition occurs at cross section A_m. Then by Bernoulli's equation:

$$\frac{1}{2}u^2 + c_s^2 \ln \rho = \frac{1}{2}c_s^2 + c_s^2 \ln \rho|_{A_m} \tag{5.36}$$

and by continuity:

$$\rho u A = \rho|_{A_m} c_s A_m \tag{5.37}$$

and combining,

$$u^2 = c_s^2 \left[1 + 2\ln \frac{\rho|_{A_m}}{\rho}\right] = c_s^2 \left[1 + 2\ln \frac{uA}{c_s A_m}\right] \tag{5.38}$$

Hence given $A(z), A_m$ and the sound speed, we may in principle solve for u using the above equation.

5.3.3 Polytropic De Laval nozzle

The equation of state is $P = K\rho^{1+1/n}$. Note that the sound speed will vary along the tube:

$$c_s^2 = \frac{n+1}{n} K \rho^{1/n} \tag{5.39}$$

$$\int \frac{dP}{\rho} = \int \frac{dP}{d\rho}\frac{d\rho}{\rho} \tag{5.40}$$

$$= \int K \frac{n+1}{n} \rho^{1/n} \frac{d\rho}{\rho} \tag{5.41}$$

$$= n c_s^2 \tag{5.42}$$

Now by continuity,

$$\dot{M} = \rho|_{A_m} c_s|_{A_m} c_s \tag{5.43}$$

and combining,

$$\rho|_{A_m} = \left[\left(\frac{\dot{M}}{A_m}\right)^2 \frac{n}{K(n+1)}\right]^{n/(2n+1)} \tag{5.44}$$

and including the Bernoulli equation,

$$\left(\frac{1}{2} + n\right) \frac{n+1}{n} K \rho^{1/n}|_{A_m} = \frac{1}{2}\left(\frac{\dot{M}}{A\rho}\right)^2 + K(n+1)\rho^{1/n} \tag{5.45}$$

which can be used to formally solve for ρ.

Chapter 6

Week 6

6.1 Friday 19 Feb 2016

6.1.1 Spherical accretion

Assume that the gas is at rest at infinity. Hence we expect the gas to accelerate from the subsonic regime into the supersonic regime near the star (treat as as point mass). Consider a barotropic equation of state and consider the steady state. Then by continuity:

$$\dot{M} = \rho u A = \rho(r) u(r) 4\pi r^2 \tag{6.1}$$

$$\implies \frac{d\ln\rho}{dr} + \frac{d\ln u}{dr} + \frac{d\ln r^2}{dr} = 0 \tag{6.2}$$

$$\implies \frac{d\ln\rho}{dr} = -\frac{d\ln u}{dr} - \frac{2}{r} \tag{6.3}$$

and by the momentum equation:

$$u\frac{du}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2} \tag{6.4}$$

$$\implies u^2\frac{d\ln u}{dr} = -\frac{1}{\rho}\frac{dP}{d\rho}\frac{d\rho}{dr} - \frac{GM}{r^2} \tag{6.5}$$

$$\implies u^2\frac{d\ln u}{dr} = -c_s^2\frac{d\ln\rho}{dr} - \frac{GM}{r^2} \tag{6.6}$$

Combining, we obtain:

$$u^2\frac{d\ln u}{dr} = -c_s^2\left(-\frac{d\ln u}{dr} - \frac{2}{r}\right) - \frac{GM}{r^2} \tag{6.7}$$

$$\implies (u^2 - c^2)\frac{d\ln u}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2} = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right) \tag{6.8}$$

Define the sonic radius $r_s = \frac{GM}{2c_s^2}$. At the sonic radius, we observe that u has to be an extremum or cross the sound speed.

6.1.2 Isothermal spherical accretion

The temperature is constant, hence the sound speed is known and the sonic radius is well-defined. We want to solve for $\rho(r)$ and $\dot{M}$. By the Bernoulli equation, we have:

$$H = \frac{1}{2}u^2 + \int \frac{dP}{\rho} + \Psi, \quad \int \frac{dP}{\rho} = c_s^2 \ln \rho \tag{6.9}$$

$$\implies \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = \frac{1}{2}c_s^2 + c_s^2 \ln \rho|_{r_s} - \frac{GM}{r_s} \tag{6.10}$$

$$\implies u^2 = 2c_s^2 \left[\ln \frac{\rho_s}{\rho} - \frac{3}{2}\right] + \frac{2GM}{r} \tag{6.11}$$

Hence we observe that close to the star, for small r, the velocity field is dominated by the effect of the point mass and the pressure does not contribute significantly. The gas is in free-fall. Now consider the limit as $r \to \infty$. Since we require that the velocity of the gas vanish at infinity, we hence require that:

$$\ln \frac{\rho_s}{\rho(\infty)} - \frac{3}{2} = 0 \tag{6.12}$$

$$\implies \rho_s = \rho(\infty)e^{3/2} \tag{6.13}$$

6.1.3 Polytropic spherical accretion

Now the sound speed is a function of the radius. Now the energy equation becomes:

$$\frac{1}{2}u^2 + c_s^2|_r \ln \rho - \frac{GM}{r} = \frac{1}{2}c_s^2 + c_s^2 \ln \rho_s - \frac{GM}{r_s} \tag{6.14}$$

Note that for a polytropic equation of state, the sound speed is a function of the density:

$$c_s^2 = \frac{n+1}{n} K \rho_s^{1/n} \tag{6.15}$$

The sound speed as a function of the density at the sonic point is:

$$c_s = \left(\frac{GM}{2}\right)^{2/3} \left(\frac{4\pi\rho_s}{\dot{M}}\right)^{1/3} \tag{6.16}$$

Implementing the Bernoulli equation, continuity and the boundary condition at infinity, when the dust settles we have:

$$c_{s,\infty}^2 = \frac{n-3/2}{n} c_s^2 \tag{6.17}$$

$$\dot{M} = \frac{\pi G^2 M^2 \rho_\infty}{c_{s,\infty}^3} \left(\frac{n}{n-3/2}\right)^{n-3/2} \tag{6.18}$$

Note that when $n = 3/2$, or $\gamma = 5/3$, the solution for $\dot{M}$ blows up. This is because the sound speed has to go to infinity at the sonic radius, and the infalling gas never actually experiences the sonic transition.

The accretion rate can also be written as (Bondi-Hoyle-Littleton accretion):

$$\dot{M} = \frac{4\pi G^2 M^2 \rho_\infty}{(c_{s,\infty}^2 + V_\infty^2)^{3/2}} \tag{6.19}$$

6.2 22 Feb 2016, Monday

6.2.1 Fluid Instabilities

Consider a fluid in the steady state. Consider a small perturbation in the fluid. There are two possibilities. The first is that the perturbation decays in time or oscillates. The second case is that the perturbation grows in time.

6.2.2 Convective instabilities and the Schwarzschild stability criterion

Assume ideal gas, hydrostatic equilibrium and uniform gravitational field. Consider a parcel of fluid with initial conditions ρ, P. Perturb the parcel by moving the parcel δz in the positive z direction antiparallel to the gravitational field. Then the fluid element is now embedded in a medium with parameters ρ', P'. The perturbed parcel will have a different density ρ^* as it expands to match the pressure outside P'. There are two cases: $\rho^* > \rho', \rho^* < \rho'$. The former corresponds to unstable rising motion and the latter corresponds to a stable return to equilibrium.

Consider the adiabatic parcel. Then to first order:

$$\rho^* = \rho\left(\frac{P'}{P}\right)^{1/\gamma} \tag{6.20}$$

$$= \rho\left(1 + \frac{1}{P}\frac{dP}{dz}\delta z\right)^{1/\gamma} \tag{6.21}$$

$$\approx \rho + \frac{1}{\gamma P}\frac{dP}{dz}\delta z \tag{6.22}$$

$$\rho' \approx \rho + \frac{d\rho}{dz}\delta z \tag{6.23}$$

Comparing, we find that under the unstable condition is:

$$\frac{\rho}{P\gamma}\frac{dP}{dz} < \frac{d\rho}{dz} \tag{6.24}$$

Suppose that the fluid is isentropic, so $K = \frac{P}{\rho^\gamma}$ is a constant. Taking its derivative with respect to z-distance and requiring $dK/dz = 0$, we observe that

$$\frac{\rho}{P\gamma}\frac{dP}{dz} = \frac{d\rho}{dz} \tag{6.25}$$

and hence if K is a constant, the fluid is stable at the margin. We may re-write the instability criterion by replacing the density derivative by a temperature dependence using the ideal gas equation:

$$-\left(1 - \frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dz} + \frac{\rho}{T}\frac{dT}{dz} < 0 \tag{6.26}$$

$$\implies \left|\frac{dT}{dz}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{dP}\left|\frac{dP}{dz}\right| \tag{6.27}$$

where the moduli are included because the first derivatives are actually negative. This is called the **Schwarzschild stability criterion**.

Observe that if $\rho^* < \rho'$ and since the pressures of the fluid parcel and the surrounding medium are equal, we have that $T^* > T'$, and the parcel is warmer than its surroundings. The parcel will hence transfer heat energy to its surroundings, and will finally cool to sink back. Repeating the same argument, the parcel will now be cooler than its surroundings, allowing the surrounding medium to heat up the parcel and repeating the cycle again. This flow forms convective cells.

6.2.3 Equation of motion of parcels in convective cells

$$\rho^* \frac{d^2(\delta z)}{dt^2} = -g(\rho^* - \rho') \tag{6.28}$$

In the limit of small perturbations, we write $\rho^* = \rho + \delta\rho$, and use the Schwarzchild equation relating the derivatives of pressure and temperature:

$$(\rho + \delta\rho)\frac{d^2(\delta z)}{dt^2} = -g\left(\frac{\rho}{T}\frac{dT}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dz}\right)\delta z \tag{6.29}$$

$$\implies \frac{d^2(\delta z)}{\delta t^2} = -g\left(\frac{1}{T}\frac{dT}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{1}{P}\frac{dP}{dz}\right)\delta z = -N^2\delta z \tag{6.30}$$

$$N^2 \equiv g\left(\frac{1}{T}\frac{dT}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{1}{P}\frac{dP}{dz}\right) \tag{6.31}$$

Observe that this equation of motion is an oscillation with angular frequency N. Hence these are internal density oscillations.

6.2.4 Jeans instability

Include gravity. Consider a uniform medium (Lagrangian and Eulerian perturbations are the same). Consider the mass conservation equation, momentum equation and Poisson equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0 \tag{6.32}$$

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla P - \nabla\Psi \tag{6.33}$$

$$\nabla^2\Psi = 4\pi G\rho \tag{6.34}$$

Substituting the perturbations into these equations and skipping all the grungy algebra (taking linearization to first order):

$$\frac{\partial(\Delta\rho)}{\partial t} + \rho_0\nabla\cdot(\Delta\boldsymbol{u}) = 0 \tag{6.35}$$

$$\frac{\Delta\boldsymbol{u}}{\partial t} = -c_s^2\nabla(\Delta\rho)\frac{1}{\rho_0} - \nabla(\Delta\Psi) \tag{6.36}$$

$$\nabla^2(\Delta\Psi) = 4\pi G\Delta\rho \tag{6.37}$$

Consider the oscillatory ansatz:

$$\Delta\rho = \rho_1 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{6.38}$$

$$\Delta\boldsymbol{u} = \boldsymbol{u}_1 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{6.39}$$

$$\Delta\Psi = \Psi_1 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{6.40}$$

This gives the Fourier space representation of the dispersion relation:

$$\omega^2 = c_s^2\left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right) = c_s^2\left(k^2 - k_J^2\right) \tag{6.41}$$

where we define the Jeans' wavenumber k_J. Now this corresponds to the boundary between imaginary and real solutions for ω, giving the distinction between the oscillatory and exponential growth modes of the perturbation. Hence if $k < k_J$, we have that ω is imaginary and the solution is unstable.

Also define the Jeans' length and the Jeans mass:

$$\lambda_J = \frac{2\pi}{k_J} \tag{6.42}$$

$$M_J = \rho_0 \lambda_J^3 \tag{6.43}$$

6.3 Wednesday 24 Feb 2016

6.3.1 2D Fluid instabilities

Consider the 2D xz plane, with the half-planes above and below the $z = 0$ plane having two different conditions. Let the fluid below the $z = 0$ plane have density and x-velocity (parallel to the $z = 0$ plane) ρ, U, and that above the $z = 0$ plane have ρ', U'. Let a uniform gravitational field act in the $-\hat{z}$ direction. Observe that a pressure gradient is necessary to ensure that the fluid remains in a steady state. Let the pressure of the fluid across the interface be continuous. Let the fluid be incompressible ($\nabla \cdot \boldsymbol{u} = 0$) and ideal. Let the flow be irrotational initially (and hence be irrotational for all time). Then it will suffice to solve for the velocity scalar potential:

$$-\nabla\Phi = \boldsymbol{u} \tag{6.44}$$

Consider a sinusoidal perturbation of the z-position of the interface: $\xi(x, t)$. We require that the boundary conditions of the problem at $z \to \pm\infty$ be that the fluid retains its initial ρ, U far away from the perturbed interface.

Now consider the momentum equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla P + \boldsymbol{g} \tag{6.45}$$

$$\implies \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\frac{1}{2}\boldsymbol{u}^2\right) = -\frac{1}{\rho}\nabla P + \boldsymbol{g}, \quad \nabla \times \boldsymbol{u} = 0 \tag{6.46}$$

$$\implies -\nabla\frac{\partial \Phi}{\partial t} + \nabla\left(\frac{1}{2}\boldsymbol{u}^2\right) = -\nabla\left(\frac{P}{\rho}\right) - \nabla\Psi \tag{6.47}$$

We may now integrate both sides with respect to space to require that:

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2 + \frac{P}{\rho} + \Psi = f \tag{6.48}$$

where f is a constant of integration.

Now we note that we may perturb the x-velocity of the half-planes by perturbing the velocity scalar potential with functions ϕ, ϕ':

$$\Phi = Ux + \phi \tag{6.49}$$

$$\Phi' = -U'x + \phi' \tag{6.50}$$

Now we require that the fluid be incompressible, so $\nabla \cdot \boldsymbol{u} = 0$ and:

$$\nabla^2 \phi = 0 \tag{6.51}$$
$$\nabla^2 \phi' = 0 \tag{6.52}$$

Note further that far away from the interface, we can require that ϕ, ϕ' vanish so that $\Phi = Ux, \Phi = U'x$ and hence the velocity field far away is the same as that of the unperturbed initial system.

The change in the velocity scalar potential perturbation with z-position is related to the displacement (to first order):

$$-\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \frac{D\xi}{Dt} = \frac{\partial \xi}{\partial t} + U\frac{\partial \xi}{\partial t} \tag{6.53}$$
$$-\left.\frac{\partial \phi'}{\partial z}\right|_{z=0} = \frac{D\xi}{Dt} = \frac{\partial \xi}{\partial t} + U'\frac{\partial \xi}{\partial t} \tag{6.54}$$

Consider the exponential ansatz $\xi(x,t) = Ae^{i(kx-\omega t)}, \phi = Ce^{i(kx-\omega t)} + jz, \phi' = C'e^{i(kx-\omega t)} + j'z$. Note that we add the jz term to account for the velocity component along the z-axis. The vanishing of the Laplacian of ϕ, ϕ' gives:

$$-k^2 + j^2 = 0 \tag{6.55}$$
$$-k^2 + (j')^2 = 0 \tag{6.56}$$

The sign of j is determined by the boundary conditions. Now we know that $\phi \to 0$ as $z \to -\infty$ and $\phi' \to 0$ as $z \to \infty$ so that the velocity field attains the unperturbed value. This gives:

$$j = k \tag{6.57}$$
$$j' = -k \tag{6.58}$$

Substituting the form of ϕ, ϕ' into the equations in (6.53 – 6.54) gives:

$$-kC = -i\omega A + UikA \tag{6.59}$$
$$kC' = i(kU' - \omega)A \tag{6.60}$$

Now we have too many unknowns. We appeal to the momentum equation (suitably rearranged and taking the gravitational potential at the perturbed interface to be $\Psi = g\xi$):

$$P = -\rho\left(-\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + g\xi\right) + \rho f \tag{6.61}$$
$$P' = -\rho'\left(-\frac{\partial \phi'}{\partial t} + \frac{(u')^2}{2} + g\xi\right) + \rho' f' \tag{6.62}$$

Since we claimed that the pressure is continuous across the interface, we equate these two expressions to obtain:

$$-\rho\left(-\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + g\xi\right) + \rho f = -\rho'\left(-\frac{\partial \phi'}{\partial t} + \frac{(u')^2}{2} + g\xi\right) + \rho' f' \tag{6.63}$$
$$\implies -\rho\left(-\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + g\xi\right) = -\rho'\left(-\frac{\partial \phi'}{\partial t} + \frac{(u')^2}{2} + g\xi\right) + K, \quad K \equiv \rho' f' - \rho f \tag{6.64}$$

Far away from the interface, $|z| \gg 0$, we may set $u \to U, u' \to U', \phi = \phi' = \xi \to 0$, which gives us:

$$\frac{1}{2}U^2\rho = \frac{1}{2}(U')^2\rho' + K \tag{6.65}$$

and this uniquely defines K. We may also replace u^2 and $(u')^2$ in (6.64) by including its dependence on ϕ:

$$u = -\nabla\phi = U - \frac{\partial \phi}{\partial x} \tag{6.66}$$

$$u = -\nabla\phi' = U' - \frac{\partial \phi'}{\partial x} \tag{6.67}$$

$$\implies u^2 \approx U^2 - 2U\frac{\partial \phi}{\partial x} \quad \text{to first order} \tag{6.68}$$

$$\implies (u')^2 \approx (U')^2 - 2U'\frac{\partial \phi'}{\partial x} \quad \text{to first order} \tag{6.69}$$

Much messy algebra later, substituting the exponential ansatz (to move into the Fourier domain), and cancelling terms,

$$\rho i\omega C - \rho U ikC + \rho gA = \rho' i\omega C' - \rho' U' ikC' + \rho' gA \tag{6.70}$$

and simplifying the amplitudes, we obtain the dispersion relation:

$$\boxed{\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho')}$$

which can be written in terms of the phase velocity:

$$v_p = \frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}} \tag{6.71}$$

6.3.2 Surface Gravity Waves

Consider fluids that are initial at rest in the previous section. Then $U = U' = 0$. Also assume that $\rho' < \rho$ so that the lighter fluid is above the heavier fluid. Then the dispersion relation simplifies:

$$v_p = \frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'}} \tag{6.72}$$

which gives a real ω for real k (dispersive waves).

Now consider the case where $\rho' > \rho$ so that the heavier fluid is sitting on the lighter fluid. Note now that for real k, we obtain imaginary ω and hence we have exponentially growing perturbations. This is called the **Rayleigh-Taylor instability**.

Now consider $\rho > \rho'$ so that the heavier fluid is below, but consider non-zero x-velocities. The term in the square root can be negative if the wavenumber is sufficiently large:

$$k > \frac{[\rho^2 - (\rho')^2]g}{\rho\rho'(U - U')^2} \tag{6.73}$$

Then if the gravitational field is zero, the RHS vanishes, and the inequality is satisfied for all k. Hence the fluid is unstable with respect to perturbations for all spatial scales (all k). For non-zero g, then there is a cut-off spatial scale for the instability to occur. This instability is known as the **Kelvin-Helmholtz instability**.

Chapter 7

Week 7

7.1 Friday 26 Feb 2016

7.1.1 Thermal instabilities and the Field Criterion

Consider temperature perturbations (WLOG consider small increase in temperature ΔT). The unperturbed state hence corresponds to thermal equilibrium $\dot{Q}_{cool} = 0$.

Case A: Constant pressure Since the pressure perturbations will be evened out on a timescale much shorter than the thermal perturbations, we may make the assumption that the pressure is constant throughout the fluid. Then consider the cooling function change as a result of the temperature perturbation:

$$\dot{Q}_{cool} \rightarrow \dot{Q}_{cool}(0) + \left.\frac{\partial \dot{Q}_{cool}}{\partial T}\right|_p \Delta T \tag{7.1}$$

If the last term is negative, then we expect that the system will be unstable since a slight heating will result in further heating. This is a runaway process. Hence the condition for instability is:

$$\boxed{\left.\frac{\partial \dot{Q}_{cool}}{\partial T}\right|_p \Delta T < 0}$$

Case B: Non-constant pressure We ignore gravity for this analysis. Assume $\boldsymbol{u}_0 = 0, \dot{Q}_0 = 0, \nabla P_0 = 0, \nabla \rho_0 = 0$, and assume adiabatic processes $P = K\rho^\gamma$. Let the fluid have the same initial adiabatic dependence $\nabla K_0 = 0$.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{7.2}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla P \tag{7.3}$$

Then consider the pressure perturbation due to density and K fluctuations:

$$P = K\rho^\gamma \implies dP = \rho^\gamma dK + \frac{\gamma P}{\rho} d\rho \tag{7.4}$$

The ideal gas relationship allows us to find the relationship between changes in pressure and temperature:

$$P = \frac{\mathcal{R}_* \rho T}{\mu} \implies dP = \frac{P d\rho}{\rho} + \frac{\mathcal{R}_* \rho dT}{\mu} \tag{7.5}$$

Combining the equations for the pressure differentials, we obtain:

$$dK = \rho^{1-\gamma}(1-\gamma)\left[\frac{P}{\rho^2}d\rho + \frac{\mathcal{R}_*}{\mu(1-\gamma)}dT\right] \tag{7.6}$$

Note that:

$$\frac{P}{\rho^2}d\rho = -PdV \tag{7.7}$$

which can be seen by taking $\rho = 1/V$. Note further that the second term is:

$$\frac{d\xi}{dT} = C_V = -\frac{\mathcal{R}_*}{\mu(1-\gamma)} \tag{7.8}$$

hence using the first law of thermodynamics, $dQ = d\xi - pdV$, so:

$$dK = -\rho^{1-\gamma}(1-\gamma)dQ \tag{7.9}$$

$$\implies \frac{dK}{dt} = \rho^{1-\gamma}(1-\gamma)\dot{Q}_{cool} \tag{7.10}$$

Now consider the Lagrangian/Eulerian perturbations (same as the initial unperturbed state is isotropic) $\Delta\rho, \Delta\boldsymbol{u}$, which result in the perturbations $\Delta K, \Delta\dot{Q}_{cool}$. Apart from the usual continuity and momentum equations, there is the additional relation:

$$\frac{\partial \Delta K}{\partial t} = -\frac{\gamma-1}{\rho_0^{\gamma-1}}\Delta\dot{Q}_{cool} \tag{7.11}$$

The resultant perturbation in $\Delta\dot{Q}_{cool}$ can be written in terms of pressure and density dependence:

$$\Delta\dot{Q}_{cool} = \left.\frac{\partial\dot{Q}_{cool}}{\partial P}\right|_{\rho}\Delta P + \left.\frac{\partial\dot{Q}_{cool}}{\partial \rho}\right|_{P}\Delta\rho \tag{7.12}$$

Define the following parameters:

$$A^* = \frac{\gamma-1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}_{cool}}{\partial P}\right|_{\rho} \tag{7.13}$$

$$B^* = \frac{\gamma-1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}_{cool}}{\partial \rho}\right|_{P} \tag{7.14}$$

$$\implies \frac{\partial \Delta K}{\partial t} = -A^*\Delta P - B^*\Delta\rho \tag{7.15}$$

Making an exponential ansatz:

$$\Delta\rho = \rho_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}+qt} \tag{7.16}$$

$$\Delta\boldsymbol{u} = \boldsymbol{u}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}+qt} \tag{7.17}$$

$$\Delta K = K_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}+qt} \tag{7.18}$$

where q plays the role of $i\omega$ and $q > 0$ implies a perturbation that grows in time; an instability. Bashing through the math and solving for the relation between the wavenumber k and q:

$$q^3 + A^*\rho_0^{\gamma}q^2 + k^2\gamma\frac{P_0}{\rho_0}q - B^*k^2\rho_0^{\gamma} = 0 \tag{7.19}$$

This can be solved to obtain the general conditions for real and positive q roots. If $B^* > 0$, then there will be a real, positive root (note that all three roots cannot be negative because the last term in the cubic will be positive in that case). Hence the condition for instability is:

$$\frac{\gamma-1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}_{cool}}{\partial \rho}\right|_{P} > 0 \implies -\frac{\mathcal{R}_*T^2}{\mu P}\left.\frac{\partial\dot{Q}_{cool}}{\partial T}\right|_{P} > 0 \implies \left.\frac{\partial\dot{Q}_{cool}}{\partial \rho}\right|_{P} < 0 \tag{7.20}$$

which is the same condition as in the constant pressure case. This condition is known as the **Field criterion**.

7.1.2 Example: Bremmstrahlung

Parametrize the cooling function as:

$$\dot{Q}_{cool} = A\rho T^{\alpha} - H \tag{7.21}$$

where the cooling component has temperature dependence while the heating component proceeds at a constant rate. Evaluating the Field Criterion:

$$\left.\frac{\partial \dot{Q}_{cool}}{\partial T}\right|_P = \frac{(\alpha - 1)AP\mu}{\mathcal{R}_*} T^{\alpha-2} \tag{7.22}$$

and we observe that the thermal instability occurs when:

$$\alpha < 1 \tag{7.23}$$

For Bremsstrahlung, $\alpha = 1/2$, so the fluid will be thermally unstable.

7.2 Monday, 29 Feb 2016

7.2.1 Viscous flows

The continuity equation still holds:

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho u_j) = 0 \tag{7.24}$$

We update the momentum equation:

$$\frac{\partial \rho u_i}{\partial t} = -\partial_j \sigma_{ij} + \rho g_i \tag{7.25}$$

where $g_i = -\partial_i \Psi$. The stress tensor is:

$$\sigma_{ij} = \rho u_i u_j + P\delta_{ij} + \sigma'_{ij} \tag{7.26}$$

where we add the extra σ'_{ij} term to represent the contribution from the viscous terms. Call it the **viscous stress tensor**.

7.2.2 Shear stress thought experiment - Molecular viscosity

Consider linear shearing flow in the ith direction. Let there be a velocity gradient in the j direction. Then due to the differential flow velocities, there will be a momentum flux cross individual fluid layers given by $\rho u_i v_j$ where v_j is the velocity of the random motions of the fluid molecules. We expect that v_j will be proportional to $\sqrt{k_B T/m}$. The net momentum flux into a streamline is given by the contributions from the streamlines immediately above and below it:

$$\mathcal{F} = \rho[u_i - (u_i + \partial_j u_l \delta l)]\alpha\sqrt{\frac{k_B T}{m}} = -\rho(\partial_j u_i)\delta l \alpha\sqrt{\frac{k_B T}{m}} \tag{7.27}$$

The separation between streamlines δl should be on the order of the mean free path $\lambda = \frac{1}{n\sigma}$. Approximating the molecules as hard spheres, we set $\sigma = \pi a^2$. Then, letting ρ be the mass density of the molecules, we obtain:

$$\delta l = \frac{m}{\rho \pi a^2} \tag{7.28}$$

$$\implies \mathcal{F} = -\rho(\partial_j u_i)\frac{m}{\rho\pi a^2}\alpha\sqrt{\frac{k_B T}{m}} \tag{7.29}$$

Substituting this ansatz for the molecular viscosity into the momentum equation, and noting that the stress tensor components are precisely the momentum flux, we obtain:

$$\frac{\partial(\rho u_i)}{\partial t} = -\partial_j(\rho u_i u_j + P\delta_{ij}) + \partial_j\left[\frac{\alpha}{\pi a^2}\sqrt{mk_B T}(\delta_j u_i)\right] + \rho g_i \tag{7.30}$$

We take α to be $\frac{1}{2}$ since it is on order unity. Call $\eta = \frac{\alpha}{\pi a^2}\sqrt{mk_B T}$, the **shear viscosity**. Observe that the shear viscosity is independent of density.

7.2.3 Navier-Stokes Equation

Write the viscosity tensor as:

$$\sigma'_{ij} = -\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) - \zeta\delta_{ij}\partial_k u_k \tag{7.31}$$

The first term proportional to η corresponds to the shear viscosity. The second term is called the bulk viscosity contribution which is due to compression.

For an isotropic substance, we require $\sigma'_{ij} = \sigma'_{ji}$. Note further that the terms along the diagonal will only contain bulk viscosity terms because the term in the parenthesis will vanish.

The momentum equation becomes:

$$\frac{\partial(\rho u_i)}{\partial t} = -\partial_j\rho u_i u_j - \partial_j P\delta_{ij} + \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i \tag{7.32}$$

Using the continuity equation to combined the LHS term and the first term on the RHS, we obtain the Navier-Stokes equation in component form:

$$\boxed{\rho\left(\frac{\partial u_i}{\partial t} + u_j\partial_j u_i\right) = -\partial_j P\delta_{ij} + \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i}$$

In vector form, and assuming isothermal, incompressible conditions, we have:

$$\boxed{\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P - \nabla\Psi + \frac{\eta}{\rho}[\nabla^2\boldsymbol{u} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{u})]}$$

η/ρ is called the kinematic viscosity ν.

Fully ionized plasma example - Braginskii-Spitzer shear visoscity The main interactions are Coulombic. The mean free path varies as T^2, so the shear viscosity coefficient goes as $\eta \propto \sqrt{m}T^{5/2}$. This form is known as the Braginskii-Spitzer shear viscosity.

7.3 Wednesday 2 March 2016

7.3.1 Vorticity in viscous flows

Make the following assumptions: $\zeta = 0, \eta =$ constant. Assume fluid is barotropic $P = P(\rho)$. Then the Navier-Stokes equation gives:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P - \nabla\Psi + \frac{\eta}{\rho}[\nabla^2\boldsymbol{u} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{u})] \tag{7.33}$$

Recall that the definition of vorticity is $\boldsymbol{w} = \nabla \times \boldsymbol{u}$. Then take the curl of the entire Navier-Stokes equation. Examine the curl of each of the terms

$$\nabla \times (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla \times \left(\frac{1}{2}\boldsymbol{u}^2 - \boldsymbol{u} \times (\nabla \times \boldsymbol{u} \right) = -\nabla \times (\boldsymbol{u} \times \boldsymbol{w}) \tag{7.34}$$

$$\nabla \times \nabla \Psi = 0 \tag{7.35}$$

$$\nabla \times \left(\frac{1}{\rho} \nabla P \right) = \nabla \left(\frac{1}{\rho} \right) \times \nabla P + \frac{1}{\rho} \nabla \times \nabla P = \left(-\frac{1}{\rho^2} \nabla \rho \right) \times \nabla P + \frac{1}{\rho} \nabla \times \nabla P = 0 \tag{7.36}$$

Hence we have:

$$\frac{\partial \boldsymbol{w}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{w}) + \eta \nabla \times \frac{1}{\rho} \nabla^2 \boldsymbol{u} \tag{7.37}$$

where we note that $\nabla \times \nabla \boldsymbol{u}^2 = 0$. Note that the first two terms correspond to Kelvin's vorticity theorem. We assume that the kinematic viscosity $\nu = \eta/\rho$ is constant. Since the viscosity is already a constant, we require that the density also be a constant. The viscous term can be evaluated explicitly:
Hence we have:

$$\boxed{\frac{\partial \boldsymbol{w}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{w}) + \frac{\eta}{\rho} \nabla^2 \boldsymbol{w}}$$

Observe that the presence of the viscous term allows vorticity to be generated even if the initial vorticity is zero. Vorticity can also be damped by the viscous term.

7.3.2 Energy dissipation in incompressible viscous flows

Consider an incompressible fluid $(\partial_i u_i) = 0$ with total kinetic energy:

$$E_{kin} = \int \frac{1}{2} \rho u^2 dV \tag{7.38}$$

Ignore gravity.
Consider the time derivative:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) = u_i \partial_t (u_i \rho) \tag{7.39}$$

$$= -u_i \partial_j \sigma_{ij} \tag{7.40}$$

$$= -u_i \partial_j (\rho u_i u_j) - u_i \partial_i \delta_{ij} P - u_i \partial_j \sigma'_{ij} \tag{7.41}$$

$$\boxed{\text{then some magic happens...}}$$

$$= -\partial_i \left[\rho u_i \left(\frac{1}{2} u^2 + \frac{P}{\rho} \right) + u_j \sigma'_{ij} \right] + \sigma'_{ij} \partial_j u_i \tag{7.42}$$

Several important magic formulae include:

$$\frac{1}{2} \partial_j (\rho u_j u_i u_i) = \frac{\rho}{2} u_i u_i \partial_j u_j + u_j \partial_j (\frac{\rho}{2} u_i u_i) = u_j u_i \partial_j (\rho u_i) \tag{7.43}$$

$$\sigma'_{ij} = \sigma'_{ji} \tag{7.44}$$

To obtain the rate of change of total kinetic energy, we integrate over the entire fluid:

$$\frac{\partial E_{kin}}{\partial t} = -\int \partial_i \left[\rho u_i \left(\frac{1}{2} u^2 + \frac{P}{\rho} \right) + u_j \sigma'_{ij} \right] dV + \int \sigma'_{ij} \partial_j u_i dV \tag{7.45}$$

$$= -\oint_S \left[\rho \boldsymbol{u} \left(\frac{1}{2} \boldsymbol{u}^2 + \frac{P}{\rho} \right) + \boldsymbol{u} \cdot \boldsymbol{\sigma}' \right] \cdot d\boldsymbol{a} + \int \sigma'_{ij} \partial_j u_i dV \tag{7.46}$$

Observe that the first surface integral corresponds to the work done by the ram pressure, thermal pressure and $\boldsymbol{\sigma'}$ over the surface. We implement the condition that the velocity vanishes at infinity. Then considering all space, the first surface integral vanishes. The second volume term corresponds to the rate of dissipation of kinetic energy due to viscous forces. Hence we have:

$$\frac{\partial E_{kin}}{\partial t} = \int \sigma'_{ij}\partial_j u_i dV \tag{7.47}$$

$$= \frac{1}{2}\int \sigma'_{ij}(\partial_j u_i + \partial_i u_j)dV, \quad \text{symmetric matrix} \tag{7.48}$$

$$= -\frac{1}{2}\int \eta\left(\partial_j u_i + \partial_i u_j\right)^2 dV, \quad \sigma'_{ij} = -\eta\left(\partial_j u_i + \partial_i u_j\right) \tag{7.49}$$

Observe that the integrand is negative definite. Hence the kinetic energy decreases monotonically. In the absence of viscosity, kinetic energy is conserved. Note that this viscous effect takes place in the shock, thereby changing the energy and entropic content of fluid elements crossing the shock.

7.3.3 Example: Viscous flow in pipe

Consider a pipe with radius R_0. Let the flow be steady and in the z direction: $\boldsymbol{u} = (0, 0, u_z)$ in cylindrical coordinates. Let the fluid be incompressible and neglect gravity. By symmetry, u_z varies with the axial radius and has to vanish at the edge of the pipe. The Navier-Stokes equation in this case is:

$$0 = -\frac{1}{\rho}\nabla P + \nu\left[\nabla^2 \boldsymbol{u}\right] \tag{7.50}$$

$$\implies \frac{1}{\rho}\frac{dP}{dz} = \nu\frac{1}{R}\frac{d}{dR}\left(R\frac{du_z}{dR}\right) \tag{7.51}$$

Since the LHS and RHS are functions of different coordinates, they must individually be constants. This implies that the gradient of pressure is a constant. Define ΔP to be the *decrease* in pressure across the pipe (this adds a minus sign).

$$\frac{1}{\rho}\frac{\Delta P}{l} = -\nu\frac{1}{R}\frac{d}{dR}\left(R\frac{du_z}{dR}\right) \tag{7.52}$$

Integrating and implementing the boundary conditions at the pipe circumference:

$$u = \frac{\Delta P}{4\nu\rho l}(R_0^2 - R^2) \tag{7.53}$$

The mass flux through the pipe can be obtained through integrating the cross section:

$$J = \int_0^{R_0} 2\pi\rho u R dR = \frac{\pi\Delta P}{8\eta}R_0^4 \tag{7.54}$$

Observe that in the limit $\nu \to 0$, the flow rate is unbounded for finite pressure drop. This means that in inviscid flows, there can be no pressure gradient.

Chapter 8

Week 8

8.1 Friday 4 Mar 2016

8.1.1 Accretion Disks

Let the objects in the disk move with Keplerian angular velocity $\Omega = \sqrt{\frac{GM}{R^3}}$ which arises from the balance of centrifugal and gravitational forces. Observe that the angular velocity varies with radius, indicating that concentric layers are shearing against each other. The shear flow results in momentum transfer towards the center of the disk, and this corresponds to mass transfer as well. Proceed in cylindrical coordinates (R, ϕ, z) and assume that the disk is axisymmetric: $\frac{df}{d\phi} = 0$ for all parameters $f(R, \theta, z)$. Assume hydrostatic equilibrium along the z-axis: $u_z = 0$. Note that the azimuthal velocity u_ϕ will be very close to the Keplerian angular velocity and will dominate the radial velocity u_R. Also assume that the bulk viscosity $\zeta = 0$ so we do not have any shocks.

Then the Eulerian continuity equation gives:

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\rho u_\phi) + \frac{\partial(\rho u_z)}{\partial z} = 0 \tag{8.1}$$

$$\implies \frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) = 0 \tag{8.2}$$

The Navier-Stokes equation (and not the momentum equation) gives (assuming u_R, u_ϕ do not have z-dependence):

$$\rho\left(\frac{\partial u_\phi}{\partial t} + u_R\frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}\right) = \eta\left(\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2}\right) + \frac{\partial \eta}{\partial R}\left(\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right) \tag{8.3}$$

Define the kinematic viscosity $\nu = \frac{\eta}{\rho}$ and the column density $\Sigma = \int \rho dz$. Define also the mass-average kinematic viscosity:

$$\tilde{\nu} \equiv \frac{\int \rho\nu dz}{\int \rho dz} = \frac{\int \eta dz}{\Sigma} \tag{8.4}$$

Let $\tilde{\nu}$ replace ν in the fluid equations:

$$\implies \frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0 \tag{8.5}$$

$$\Sigma\left(\frac{\partial u_\phi}{\partial t} + u_R\frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}\right) = \tilde{\nu}\Sigma\left(\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2}\right) + \frac{\partial(\tilde{\nu}\Sigma)}{\partial R}\left(\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right) \tag{8.6}$$

Combining,

$$Ru_\phi \frac{\partial \Sigma}{\partial t} + u_\phi \frac{\partial}{\partial R}(R\Sigma u_R) = \frac{1}{R}\frac{\partial}{\partial R}\left(\tilde{\nu}\Sigma R^3 \frac{d\Omega}{\partial R}\right) - R\Sigma\left(\frac{\partial u_\phi}{\partial t} + u_R \frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}\right) \tag{8.7}$$

$$\implies \frac{\partial}{\partial t}(R\Sigma u_\phi) + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\tilde{\nu}\Sigma R^3 \frac{d\Omega}{dR}\right) \tag{8.8}$$

Note that the first term corresponds to the rate of change of the angular momentum (per unit area). The second term corresponds to the advection of angular momentum radially (i.e. net rate of angular momentum loss per unit area due to advection). Advection: transfer due to mass flow. The RHS is the net torque exerted (per unit area) due to viscous processes. The net viscous torque $dG(R)$ exerted on an annulus is hence obtained by multiplying the surface area:

$$dG(R) = 2\pi R dR \frac{1}{R}\frac{\partial}{\partial R}\left(\tilde{\nu}\Sigma R^3 \frac{d\Omega}{dR}\right) \tag{8.9}$$

$$\implies G(R) = 2\pi\tilde{\nu}\Sigma R^3 \frac{d\Omega}{dR} \tag{8.10}$$

For a Keplerian orbit, the azimuthal velocity is approximately constant:

$$\frac{\partial u_\phi}{\partial t} \approx 0 \tag{8.11}$$

This allows us to solve for the radial velocity:

$$u_R = \frac{\frac{\partial}{\partial R}\left(\tilde{\nu}\Sigma R^3 \frac{d\Omega}{dR}\right)}{R\Sigma \frac{\partial}{\partial R}\left(R^2\Omega\right)} \tag{8.12}$$

and using the Keplerian angular velocity,

$$u_R = -\frac{3\frac{\partial}{\partial R}\left(\tilde{\nu}\Sigma R^{1/2}\right)}{\Sigma R^{1/2}} \tag{8.13}$$

Knowing u_R allows us to calculate the surface (column) density by plugging it into the continuity equation:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\tilde{\nu}\Sigma R^{1/2}\right)\right] \tag{8.14}$$

The above is known as the **viscous diffusion equation for the accretion disk**.

8.1.2 Analysis of solutions

Consider a constant kinematic viscosity. Let the mass of the disk be contained at some radius R_0 initially. Note that the diffusion equation will cause the density to spread out in time. However, the spreading is not symmetric; most of the mass will flow inward. However, there will be some mass flowing outward, and this mass will carry the majority of the angular momentum outward.

Define the radial viscous timescale $t_v = \frac{R}{u_R}$. It can be shown that the viscous timescale is on the order of $t_v \sim \frac{R^2}{\tilde{\nu}} = \frac{R\mathbb{R}}{u_\phi}$ so that $u_R = \frac{\tilde{\nu}}{R}$. The Reynolds number $\mathbb{R}$ is defined:

$$\mathbb{R} = \frac{Ru_\phi}{\tilde{\nu}} \tag{8.15}$$

In protostellar disks, the Reynolds number is very high: 10^{14}, so the viscous time is greater than the Hubble time. Hence the molecular viscosity does not operate in protostellar disks. There are, however, other mechanisms that implement an effective viscosity to decrease the viscous time.

8.2 Monday 7 Mar 2016

8.2.1 Steady Thin Disks

Recall the continuity equation written in terms of the column/surface density $\Sigma = \int \rho dz$:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma u_R\right) = 0 \tag{8.16}$$

For a steady disk, there are no time-dependences, which allows us to integrate once:

$$R\Sigma u_R = c_1 \tag{8.17}$$

We know the constant mass accretion rate with circular symmetry is related to the radial velocity and surface density:

$$\dot{m} = -2\pi R\Sigma u_R \tag{8.18}$$

and hence $c_1 = -\frac{\dot{m}}{2\pi}$. Under the assumption that the azimuthal velocity is equal to the Keplerian velocity, we also have a relation for the radial velocity:

$$u_R = -\frac{3\frac{\partial}{\partial R}\left(\tilde{\nu}\Sigma R^{1/2}\right)}{\Sigma R^{1/2}} \tag{8.19}$$

Hence upon substitution of (8.17), we obtain a differential equation. Impose the boundary condition that:

$$\tilde{\nu}\Sigma = 0, \quad R = R_* \tag{8.20}$$

where R_* represents the outer radius of the disk. Solving the differential equation on the domain $R < R^*$ by separation of variables, we hence obtain the dependence:

$$\tilde{\nu}\Sigma = \frac{\dot{m}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{8.21}$$

Observe that the kinematic viscosity is related to the mass inflow rate.

8.2.2 Viscous dissipation and luminosity

Define the dissipation power $F_{diss} = -\frac{\partial E_{kin}}{\partial t}$. In component form:

$$F_{diss} = -\int \sigma'_{ij}\partial_j u_i \frac{dV}{2\pi R dR} \tag{8.22}$$

$$= \frac{1}{2}\int \eta\left(\partial_j u_i + \partial_i u_j\right)^2 dz, \quad \text{substituting viscous stress tensor and symmetrizing} \tag{8.23}$$

$$= \int \eta R^2 \left(\frac{d\Omega}{dR}\right)^2 dz \tag{8.24}$$

Recall that the viscous torques go as $G(R) \propto R\frac{d\Omega}{dR}$. This is consistent with the form of equation (8.24) so that the dissipation is due to the viscous torques. Performing the integration over the z-direction, using the definition of the surface density, and substituting out the viscosity for the kinematic viscosity, we hence obtain:

$$F_{diss} = \nu\Sigma R^2\left(\frac{d\Omega}{dR}\right)^2 \tag{8.25}$$

$$= \frac{3GM}{4\pi R^3}\dot{m}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right], \quad \Omega = \sqrt{\frac{GM}{R^3}} \tag{8.26}$$

Integrating over all the disc area, we obtain the total power emitted:

$$\int_A F_{diss} = \int_{R_*}^{\infty} F_{diss} 2\pi R dR = \frac{GM\dot{m}}{2R_*} \tag{8.27}$$

Note that the total power can be interpreted as the rate of gravitational potential energy loss due to the transfer of mass inwards. However, note that the factor of 2 in the denominator implies that only half of the gravitational potential energy flow goes to the total power emitted (i.e. radiated away as luminosity). The other half of the energy flow goes to the kinetic energy of the inflowing material.

Estimating temperature of accretion disks We implement the **Optically thick limit**. This means that the radiation interacts many times with the fluid so that it attains a well-defined temperature and a Maxwellian distribution. Define the effective temperature as a function of radius $T_{eff}(R)$. We match the blackbody emission flux with the viscous dissipation F_{diss}. There is a factor of 2 because the area of the disk has a top and bottom surface:

$$\sigma_B T_{eff}(R)^4 = \frac{3GM\dot{m}}{4\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{8.28}$$

In the limit of large radii, we neglect the R-dependent term in the parenthesis and hence obtain:

$$T_{eff} \sim R^{-3/4} \tag{8.29}$$

which is called the **characteristic power law** for disks.

We may also obtain the total radiated flux at frequency f using the Planck distribution:

$$F(f) = \int_{R_*}^{R_{out}} \frac{2h}{c^2} \frac{f^3}{e^{hf/k_B T_{eff}} - 1} 2\pi R dR \tag{8.30}$$

8.2.3 Plasmas and Magnetohydrodynamics

Assume that we can describe the motion of charged particles using fluid equations. That is, assume that we can define a fluid element and that the particles collide. Consider positively and negatively charged particles with masses and velocities $m_+, m_-, \boldsymbol{u}_+, \boldsymbol{u}_-$. Mass conservation requires:

$$\frac{\partial n_+}{\partial t} + \nabla \cdot (n_+ \boldsymbol{u}_+) = 0 \tag{8.31}$$

$$\frac{\partial n_-}{\partial t} + \nabla \cdot (n_- \boldsymbol{u}_-) = 0 \tag{8.32}$$

Define the center of mass velocity and total density:

$$\boldsymbol{u} = \frac{m_+ n_+ \boldsymbol{u}_+ + m_- n_- \boldsymbol{u}_-}{m_+ n_+ + m_- n_-} \tag{8.33}$$

$$\rho = m_+ n_+ + m_- n_- \tag{8.34}$$

Then we may combine the two mass conservation equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{8.35}$$

Also define the total charge and current densities:

$$\mathcal{Q} = n_+ e^+ + n_- e^- \tag{8.36}$$

$$\boldsymbol{j} = e^+ n_+ \boldsymbol{u}_+ + e^- n_- \boldsymbol{u}_- \tag{8.37}$$

so that charge conservation can be written as:

$$\frac{\partial \mathcal{Q}}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \tag{8.38}$$

Now consider the Lorentz force:

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \tag{8.39}$$

The momentum equation for each species will then have to include the Lorentz force:

$$m_+ n_+ \left(\frac{\partial \boldsymbol{u}_+}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_+ \right) = e^+ n_+ \left(\boldsymbol{E} + \boldsymbol{u}_+ \times \boldsymbol{B} \right) - f_+ \nabla P \tag{8.40}$$

$$m_- n_- \left(\frac{\partial \boldsymbol{u}_-}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_- \right) = e^- n_- \left(\boldsymbol{E} + \boldsymbol{u}_- \times \boldsymbol{B} \right) - f_- \nabla P \tag{8.41}$$

where $f_\pm$ is the fraction of the pressure applied onto the positive and negative particles respectively. Note that the $\boldsymbol{u}$ in the $\boldsymbol{u} \cdot \nabla \boldsymbol{u}_+$ term refers to the center of mass velocity. The combined momentum equation can be written:

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = \mathcal{Q}\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} - \nabla P \tag{8.42}$$

8.3 Wednesday, 9 March 2016

8.3.1 Magnetohydrodynamics and Maxwell's Equations

We rewrite the continuity equation in terms of the current and charge densities:

$$\frac{\partial \mathcal{Q}}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \tag{8.43}$$

The momentum equation is:

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = \mathcal{Q}\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} - \nabla P \tag{8.44}$$

To form a closed set of equations, we introduce Ohm's Law:

$$\boldsymbol{j} = \sigma \left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \right) \tag{8.45}$$

where σ is the electrical conductivity. We also include Maxwell's equations:

$$\nabla \cdot \boldsymbol{B} = 0 \tag{8.46}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\mathcal{Q}}{\epsilon_0} \tag{8.47}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \tag{8.48}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{8.49}$$

where the speed of light is $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

8.3.2 Non-relativistic treatment of plasmas

First consider the dimensions of the parameters. The ratio E/B has units of velocity, which we consider to be the characteristic velocity. Observe that we may compare the sizes of the displacement current contribution to that of the curl of the magnetic field:

$$\frac{\left|\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right|}{|\nabla \times \boldsymbol{B}|} \sim \frac{u^2}{c^2} \tag{8.50}$$

In the non-relativistic limit, we consider that the characteristic velocity $\boldsymbol{u}$ is much smaller than the speed of light, and hence this ratio is small. We hence neglect the displacement current contribution and write an approximate equation:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{8.51}$$

We also consider that the curl equations give (approximately):

$$\frac{E}{l} \sim \frac{\mathcal{Q}}{\epsilon_0} \tag{8.52}$$

$$\frac{B}{l} \sim \mu_0 j \tag{8.53}$$

$$\implies \frac{\mathcal{Q}}{\epsilon_0 \mu_0 j} \sim \frac{E}{B} \tag{8.54}$$

$$\implies \frac{|\mathcal{Q}\boldsymbol{E}|}{|\boldsymbol{j} \times \boldsymbol{B}|} \sim \frac{u^2}{c^2} \ll 1 \tag{8.55}$$

$$\implies \mathcal{Q} \ll \frac{j}{u} \tag{8.56}$$

We are hence justified in assuming charge neutrality.

8.3.3 Flux freezing approximation

Assume that the electrical conductivity σ is a constant in time and space. Take the curl of the magnetic field curl equation and simplify using Ohm's Law to get:

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla \times (\mu_0 \boldsymbol{j}) \tag{8.57}$$

$$= \mu_0 \sigma \left[\nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})\right] \tag{8.58}$$

$$= \mu_0 \sigma \left[-\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})\right] \tag{8.59}$$

$$= \mu_0 \sigma \left[-\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})\right] \tag{8.60}$$

$$\implies \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B} \tag{8.61}$$

Note that this equation looks similar to Helmholtz's vorticity equation. The second term hence corresponds to the advection of the magnetic field lines by the fluid. The third term plays the role of the "viscosity" term and diffuses the magnetic field lines in space. Hence the finite electrical conductivity (in a non-perfect conductor) allows for the diffusion of the magnetic field lines. We may, however, assume that the plasma is a good conductor and take $\sigma \to \infty$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \tag{8.62}$$

Then the flux of the magnetic field will hence be conserved in co-moving surfaces with the fluid. The magnetic flux hence moves along with the fluid in a Lagrangian manner. This approximation is known as the **Flux freezing approximation**.

Another implication of the high conductivity approximation can be obtained by examining Ohm's Law:

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \tag{8.63}$$

and since $\sigma \to \infty$ but the current density $\boldsymbol{j}$ is finite, we obtain that the term in the parenthesis must vanish:

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = 0 \implies \boldsymbol{E} \cdot \boldsymbol{B} = 0 \tag{8.64}$$

where the last implication was obtained by taking the dot product of both sides of the equation with $\boldsymbol{B}$.

8.3.4 Ideal MHD equations

To summarize the previous analysis, we write the Ideal MHD equations under the non-relativistic, charge-neutral, high conductivity regime, where the electric field has been completely substituted with the magnetic field:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{8.65}$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = \boldsymbol{j} \times \boldsymbol{B} - \nabla P \tag{8.66}$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \frac{\partial B}{\partial t} \tag{8.67}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{8.68}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{8.69}$$

$$P = K\rho^{\gamma}, \quad \gamma = \frac{5}{3} \tag{8.70}$$

8.3.5 Magnetic Pressure

We proceed qualitatively to obtain a condition where the magnetic pressure is significant compared to the thermal pressure (hence requiring an MHD treatment). Recall the Lorentz force:

$$\boldsymbol{F}_L = e\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) \tag{8.71}$$

We neglect the electric field term and define a magnetic force density $\boldsymbol{f}_{mag}$ that is given by:

$$\boldsymbol{f}_{mag} = \boldsymbol{j} \times \boldsymbol{B} \tag{8.72}$$

Then by the Maxwell equation for the curl of the magnetic field, we obtain:

$$\boldsymbol{f}_{mag} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \implies \boldsymbol{f}_{mag} = \frac{1}{\mu_0}\left(-\nabla\left(\frac{B^2}{2}\right) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}\right) \tag{8.73}$$

Define the magnetic pressure corresponding to the first term on the RHS:

$$p_{mag} = \frac{B^2}{2\mu_0} \tag{8.74}$$

The second term $(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$ corresponds to the magnetic tension per unit area due to the magnetic field lines. The tension operates along the direction of magnetic field lines.

Proceed by dimensional analysis. Recall that the ram pressure went as $\frac{1}{2}\rho u^2$, the thermal pressure went as ρc_s^2 and now we have defined a magnetic pressure that goes as $\frac{B^2}{2\mu_0}$. We define a characteristic velocity associated with the pressure gradients driven by magnetic fields in the plasma by writing:

$$\frac{1}{2}\frac{B^2}{\mu_0} \sim \frac{1}{2}\rho v_A^2 \implies v_A = \sqrt{\frac{B^2}{\rho\mu_0}} \tag{8.75}$$

where we call v_A the **Alfven velocity**. We may also define the Alfven velocity as a vector pointing along the field lines:

$$\boldsymbol{v}_A = \boldsymbol{B}\frac{1}{\sqrt{\rho_0\mu_0}} \tag{8.76}$$

Note that if the magnetic pressure is high enough, it can suppress star formation by reducing the density of collapsed regions and smoothing out the shocks.

8.3.6 Plasma Waves

Consider an ideal fluid with constant initial density, uniform initial $\boldsymbol{B}_0$. Consider a small perturbation in exponential form proportional to $e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$.

Case 1: Longitudinal MHD waves $\boldsymbol{k} \perp \boldsymbol{B}$. The velocity of these waves go as $\sqrt{c_s^2 + v_A^2}$.

Case 2: $\boldsymbol{k} \parallel \boldsymbol{B}$. There are two types of solutions. The first is ordinary sound waves that propagate with velocity c_s. This corresponds to waves propagating along the field lines, and hence are not affected by the magnetic field. The second solution is associated with the magnetic tension, which requires a non-isotropic magnetic field so that the tension term is non-vanishing. These are transverse waves with particle perturbation velocities perpendicular to the Alfven velocity. We call these pure MHD waves, propagating with velocity v_A.