Ph125c Book Notes

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S: Shankar, Principles of Quantum Mechanics (2nd Ed.) C: Cooper, Khare, Sukhatme, Supersymmetry in Quantum Mechanics. P: Peskin, An Introduction to Quantum Field Theory (1st Ed), CBG: Dr David Tong, Cambridge Part III Mathematics: Quantum Field Theory notes 2006-2007.

Momentum space resolution of identity (S 21.1.13, Pg 584)

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} |p\rangle\langle p|$$

Electromagnetic Lagrangian (2.2.2, Pg 83)

$$\mathcal{L}_{EM} = \frac{1}{2} m\boldsymbol{v}\cdot\boldsymbol{v} - q\phi + \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A}$$
$$\boldsymbol{p} = m\boldsymbol{v} + \frac{q\boldsymbol{A}}{c}, \quad \text{2.2.7, Pg 84}$$

Landau Levels (S Pg 587) In the presence of a uniform magnetic field, the single particle Hamiltonian has harmonic oscillator form with canonical variables:

$$P = P_y - \frac{qBX}{2c}$$
$$Q = \frac{cP_x + qYB/2}{qB}$$

and eigenenergies:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0, \quad \omega_0 = \frac{qB}{\mu c}$$

The lowest Landau level is infinitely degenerate and can be spanned by the set of functions (S 21.1.42, Pg 589):

$$\psi_{0,m} = z^m \exp\left[-\frac{qB}{4\hbar c} zz^*\right]$$

Berry phase (S 21.1.57, Pg 593)

$$\gamma = i\int_0^t \langle n(t')|\frac{d}{dt'}|n(t')\rangle dt'$$

Berry potential (S 21.1.65, Pg 595)

$$A^n(R) = i\hbar\langle n(R)|\frac{d}{dR}|n(R)\rangle$$

where $R(t)$ is a parameter that parametrizes the time dependence of the Hamiltonian. A gauge transformation on the state vectors induces the transformation (S 21.1.67-67, Pg 595):

$$|n(R)\rangle \to e^{i\chi(R)}|n(R)\rangle$$
$$A^n(R) \to A^n(R) - \hbar\frac{d\chi}{dR}$$

The Berry potential is used to calculate the phase factor accumulated by a cyclic change (S 21.1.64, Pg 595):

$$e^{i\gamma} = \exp\left(\frac{i}{\hbar}\int_0^t A^n(R)\frac{dR}{dt'}dt'\right)$$

Coherent state (S Pg 608) is an eigenstate of the destruction operator:

$$|z\rangle = e^{za^\dagger}|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$$
$$a|z\rangle = z|z\rangle$$
$$\langle z| = \langle 0|e^{z^* a}$$
$$\langle z|a^\dagger = \langle z|z^*$$

The completeness relation is (21.1.127):

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdy}{\pi}|z\rangle\langle z|e^{-z^* z}$$

where $z = x + iy$. Under time evolution the coherent state remains a coherent state with new eigenvalue (S 21.1.131, Pg 610):

$$U(t)|z\rangle = |ze^{-i\omega t}\rangle$$

The inner product of two coherent states is (S 21.1.126, Pg 609):

$$\langle z_2|z_1\rangle = e^{z_2^* z_1}$$

From an online source:

$$a^\dagger|z\rangle = \frac{\partial}{\partial z}|z\rangle$$

Imaginary time formalism (S Pg 613)

$$t \to -i\tau$$

which solves the equation (S 21.2.3, Pg 614):

$$-\hbar\frac{d}{d\tau}|\psi(\tau)\rangle = H|\psi(\tau)\rangle$$

The propagator is (S 21.2.4):

$$U(\tau) = \sum_n |n\rangle\langle n|\exp\left(-\frac{1}{\hbar}E_n\tau\right)$$

which can be approximated semi-classically (S 21.2.22):

$$U(\tau) = \exp\left(-\frac{1}{\hbar}H\tau\right) \approx \exp\left(-\frac{1}{\hbar}S_{cl}\right)$$

Partition function (S 21.2.45, Pg 624)

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int_{-\infty}^{\infty}\langle x|e^{-\beta H}|x\rangle dx$$

Hyperbolic double angle formulae

$$\sinh 2x = 2\sinh x\cosh x$$
$$\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$$
$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

Ground state potential from ground state wavefunction (C 3.2, Pg 15) *Note that the notation here uses W(x) while Cheung's notes use W'(x)*

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)}$$

Supersymmetric operators (C 3.4, Pg 16)

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$
$$A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$
$$H = A^\dagger A$$

where we have the Riccati equation (C 3.5, Pg 16) and the superpartner potential:

$$V_1(x) = W(x)^2 - \frac{\hbar}{\sqrt{2m}}W'(x)$$
$$V_2(x) = W(x)^2 + \frac{\hbar}{\sqrt{2m}}W'(x)$$

and can relate the superpotential to the ground state wavefunction (C 4.2, Pg 36 and C 3.6, Pg 16):

$$\psi_0^{(1)} = N\exp\left[-\int^x W_1(y)dy\right]$$
$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)}$$

Energy eigenvalues of superpotential partners (C 3.12-3.14, Pg 17) The eigenequations are (C 3.9,3.11, Pg 16-17):

$$H_2(A\psi_n^{(1)}) = E_n^{(1)}(A\psi_n^{(1)})$$
$$H_1(A^\dagger\psi_n^{(2)}) = E_n^{(2)}(A^\dagger\psi_n^{(2)})$$

and introducing normalization coefficients:

$$E_n^{(2)} = E_{n+1}^{(1)}$$
$$E_0^{(1)} = 0$$
$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-1/2} A\psi_{n+1}^{(1)}$$
$$\psi_{n+1}^{(1)} = [E_n^{(2)}]^{-1/2} A^\dagger \psi_n^{(2)}$$

Note that if $E_0^{(1)} \neq 0$, then (C 3.79, Pg 31):

$$E_n^{(2)} = E_{n+1}^{(1)}$$
$$\psi_n^{(2)} = [E_{n+1}^{(1)} - E_0^{(1)}]^{-1/2} A\psi_{n+1}^{(1)}$$

In general (n systems, C 3.87, Pg 33):

$$E_n^{(m)} = E_{n+1}^{(m-1)} = \ldots = E_{n+m-1}^{(1)}$$
$$\psi_n^{(m)} = \left(\prod_{i=0}^{m-2} \frac{A_{i+1}}{\sqrt{E_{n+m-1}^{(1)} - E_i^{(1)}}}\right)\psi_{n+m-1}^{(1)}$$
$$R_m(k) = \left(\prod_{i=1}^{m-1} \frac{W_-^{(i)} - ik}{W_-^{(i)} + ik}\right) R_1(k)$$
$$T_m(k) = \left(\prod_{i=1}^{m-1} \frac{W_-^{(i)} - ik}{W_+^{(i)} - ik'}\right) T_1(k)$$
$$k \propto \sqrt{E - (W_-^{(1)})^2}$$
$$k' \propto \sqrt{E - (W_+^{(1)})^2}$$

Note that $A, A^\dagger$ change the number of nodes in the wavefunction! The ground state of system 2 has no nodes, even though its energy is equal to the first excited energy level of system 1.

Supercharges Consider the matrix SUSY Hamiltonian (C 3.15, Pg 17)

$$\boldsymbol{H} = \begin{pmatrix} \boldsymbol{H}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_2 \end{pmatrix}$$

Construct the supercharge matrices (C 3.16-17, Pg 17) :

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A} & \boldsymbol{0} \end{pmatrix}$$
$$\boldsymbol{Q}^\dagger = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{A}^\dagger \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$$

with commutation relations (C 3.18, Pg 18):

$$[H, Q] = [H, Q^\dagger] = 0$$
$$\{Q, Q^\dagger\} = H$$

Infinite well superpotential (C 3.22-26, Pg 19-20) The infinite well energies are (after displacement and relabelling)

$$E_n^{(1)} = \frac{n(n+2)\hbar^2\pi^2}{2mL^2}, \quad n = 0, 1, 2, \ldots$$

with eigenfunctions:

$$\psi_n^{(1)} = \sqrt{\frac{2}{L}} \sin\frac{(n+1)\pi x}{L}, \quad 0 \leq x \leq L$$

The supersymmetric partner potential is:

$$V_2(x) = \frac{\hbar^2\pi^2}{2mL^2}\left(2\mathrm{cosec}^2(\pi x/L) - 1\right)$$

with first few wavefunctions:

$$\psi_0^{(2)} = -2\sqrt{\frac{2}{3L}}\sin^2\frac{\pi x}{L}$$
$$\psi_1^{(2)} = -\frac{2}{\sqrt{L}}\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}$$

Scattering off superpotentials (C Pg 21) The 1D scattering eigenfunction is:

$$\psi_k(x) \to e^{ikx} + R(k)e^{-ikx}, \quad x \to -\infty$$
$$\psi_{k'}(x) \to T(k)e^{ik'x}, \quad x \to \infty$$

If the superpotential is finite at infinity:

$$W(x \to \pm\infty) = W_\pm < \infty$$

Then the wavenumbers far away satisfy:

$$k \propto \sqrt{E - W_-^2}$$
$$k' \propto \sqrt{E - W_+^2}$$

and the coefficients satisfy:

$$R_1(k) = \frac{W_- + ik}{W_- - ik} R_2(k)$$
$$T_1(k) = \frac{W_+ - ik'}{W_- - ik} T_2(k)$$

so that $|R_1|^2 = |R_2|^2, |T_1|^2 = |T_2|^2$.

Hyperbolic tangent superpotential (C 3.33, Pg 22) For:

$$W(x) = A\tanh\alpha x$$
$$V_1 = A^2 - A(A + \alpha\frac{\hbar}{\sqrt{2m}})\mathrm{sech}^2\alpha x$$
$$V_2 = A^2 - A(A - \alpha\frac{\hbar}{\sqrt{2m}})\mathrm{sech}^2\alpha x$$

Note that if $A = \alpha\frac{\hbar}{\sqrt{2m}}$, then V_2 is a constant potential. Hence V_1 is a reflectionless potential since the (magnitude squared) coefficients of reflection and transmission are the same for both systems.

Shape invariance (C 4.1, Pg 35) Two partner potentials are shape invariant if:

$$V_2(x; a_1) = V_1(x; f(a_1)) + R(a_1)$$

If this holds, the nth Hamiltonian looks like (C 4.4, Pg 36):

$$H_n = \frac{p^2}{2m} + V_1(x; f^{n-1}(a_1)) + \sum_{k=1}^{n-1} R(f^{k-1}(a_1))$$

The ground state energies are hence (C 4.5, Pg 36):

$$E_0^{(n)} = \sum_{k=1}^{n-1} R(f^{k-1}(a_1))$$

and hence the complete energy spectrum of H_1 is (C 4.6, Pg 37):

$$E_n^{(1)}(a_1) = \sum_{k=1}^{n} R(f^{k-1}(a_1))$$
$$E_0^{(1)}(a_1) = 0$$

To generate the wavefunctions (C 4.7-4.8, Pg 37):

$$\psi_n^{(1)} \propto \left(\prod_{k=1}^{n} A^\dagger(x; a_k)\right)\psi_0^{(1)}(x; a_{n+1})$$
$$\psi_n^{(1)} \propto A^\dagger(x; a_1)\psi_{n-1}^{(1)}(x; a_2)$$

where we use the ground state wavefunction for the $n+1$st potential to generate the original system eigenfunctions.
The scattering amplitudes for shape invariant potentials are:

$$R_1(k; a_1) = \left(\frac{W_-(a_1) + ik}{W_-(a_1) - ik}\right) R_1(k; a_2)$$
$$T_1(k; a_1) = \left(\frac{W_+(a_1) - ik'}{W_-(a_1) - ik}\right) T_1(k; a_2)$$

Bloch waves (W, Pg 76-77) Let the Hamiltonian be invariant under the spatial translations:

$$\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{L}_r, \quad r = 1, 2, 3$$

Then the solutions of Schrodinger's equation are Bloch waves:

$$\psi(\boldsymbol{x}) = e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\phi(\boldsymbol{x})$$
$$\phi(\boldsymbol{x} + \boldsymbol{L}_r) = \phi(\boldsymbol{x}), \quad r = 1, 2, 3$$

where $\boldsymbol{q}$ is a wave vector defined by (for $r = 1, 2, 3$):

$$\boldsymbol{q} \cdot \boldsymbol{L}_r = \theta_r$$
$$0 \leq \theta_r < 2\pi$$

Also note that ψ, ϕ satisfy the time-independent equations (note $\hbar = 1$):

$$H(\nabla, \boldsymbol{x})\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x})$$
$$H(\nabla + i\boldsymbol{q}, \boldsymbol{x})\phi(\boldsymbol{x}) = E\phi(\boldsymbol{x})$$

Bloch's Theorem statement The eigenstates ψ of the one-electron Hamiltonian with periodic potential $U(\boldsymbol{r}+\boldsymbol{R})=U(\boldsymbol{r})$ for all $\boldsymbol{R}$ in a Bravais lattice, can be chosen to have the form of a plane wave times a function with the periodicity of the Bravais lattice:

$$\psi_{nk}(\boldsymbol{r})=e^{i\boldsymbol{k}\cdot\boldsymbol{r}}u_{nk}(\boldsymbol{r})$$

Born-Von Karman Boundary condition (AM 8.22, Pg 136)

$$\psi(\boldsymbol{r}+N_i\boldsymbol{a}_i)=\psi(\boldsymbol{r}),\quad i=1,2,3$$

Delta function periodic model (AM 8.80, Pg 149) Consider a periodic potential $U(x)=\sum_n g\delta(x-na)$. Then:

$$|t|=\cos\delta$$
$$\cot\delta=-\frac{\hbar^2K}{mg}$$

Electric and magnetic fields in terms of potentials (W 10.1.12, Pg 298)

$$\boldsymbol{E}=-\frac{\dot{\boldsymbol{A}}}{c}-\nabla\phi$$
$$\boldsymbol{B}=\nabla\times\boldsymbol{A}$$

Electromagnetic Lagrangian (W 10.1.3, Pg 298)

$$L=\frac{m}{2}\dot{\boldsymbol{x}}^2-q\phi+\frac{q}{c}\dot{\boldsymbol{x}}\cdot\boldsymbol{A}$$

Electromagnetic Hamiltonian (W 10.1.9, Pg 300)

$$H=\frac{(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A})^2}{2m}+q\phi$$

Gauge transformation of Hamiltonian (W Pg 301) Make the gauge transformation:

$$\boldsymbol{A}'=\boldsymbol{A}+\nabla\alpha(\boldsymbol{x},t)$$
$$\phi'=\phi-\frac{1}{c}\frac{\partial}{\partial t}\alpha(\boldsymbol{x},t)$$

The Hamiltonian is not gauge invariant. Define the unitary operator:

$$U(t)=\exp\left[i\sum_n\frac{q_n}{\hbar c}\alpha(\boldsymbol{x}_n,t)\right]$$

Then the momentum operator can be transformed:

$$U(t)\boldsymbol{p}_n(t)U^{-1}(t)=\boldsymbol{p}_n(t)-\frac{q_n}{c}\nabla\alpha(\boldsymbol{x},t)$$

and the Hamiltonian in the new gauge is:

$$H'=UH(\boldsymbol{x},\boldsymbol{p},t)U^{-1}+i\hbar\left[\frac{d}{dt}U\right]U^{-1}$$

so that the transformed state vector:

$$\Psi'(t)=U(t)\Psi(t)$$

satisfies the time-dependent Schrodinger equation:

$$i\hbar\frac{d}{dt}\Psi'(t)=H'(t)\Psi'(t)$$

If $\boldsymbol{E},\boldsymbol{B}$ are time independent, we pick a gauge transformation that is also time independent, so $H'=UHU^{-1}$ and $\Psi'=U\Psi$ is an eigenstate of H' with the same eigenvalue E.

Periodicity of occupancy as function of magnetic field(W 10.3.16, Pg 305)

$$\Delta\left(\frac{1}{B_z}\right)=\frac{\hbar e}{m_ec\mathcal{E}_F}$$

where $\mathcal{E}_F$ is the partial Fermi energy (Fermi energy minus lowest energy eigenvalue $\mathcal{E}_0$).

Effect of vector potential on action (F 21.1, Pg 21-2) With CGS units, the vector potential adds a phase to the wavefunction:

$$e^{i\theta_1}=\exp\left(\frac{iq}{\hbar c}\int_{P_1}d\boldsymbol{r}\cdot\boldsymbol{A}\right)$$

Effect of vector potential on momentum operator (LN)

$$-i\hbar\nabla\to-i\hbar\nabla-\frac{q}{c}\boldsymbol{A}$$

where $-i\hbar\nabla=\boldsymbol{p}$, and this momentum is not the usual mv momentum.

Probability current with vector potential (F 21.12, Pg 21-4)

$$\boldsymbol{J}=\frac{1}{2}\left[\left(\frac{\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}}{m}\psi\right)^*\psi+\psi^*\left(\frac{\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}}{m}\psi\right)\right]$$
$$\implies\frac{\partial P}{\partial t}=-\nabla\cdot\boldsymbol{J}$$
$$P=\psi^*\psi$$

Multiplying Pauli Matrices (S 20.2.15, Pg 568)

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B})=\boldsymbol{A}\cdot\boldsymbol{B}+i\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B})$$

Kinetic momentum operator (S 20.2.4, Pg 567)

$$\boldsymbol{\pi}=\boldsymbol{P}-\frac{q}{c}\boldsymbol{A}$$

and the cross product (noting that it is an operator):

$$\boldsymbol{\pi}\times\boldsymbol{\pi}=\frac{iq\hbar}{c}\boldsymbol{B}$$

Natural units (P xix)

$$\hbar=c=1$$

[length]=[time]=1/[energy]=1/[mass].

QFT operators (P xx)

$$p^\mu=i\partial^\mu=i\left(\frac{\partial}{\partial x^0},\nabla\right)$$
$$E=i\frac{\partial}{\partial x_0}$$

Least action for classical field (CBG Pg 8) Given a Lagrangian density $\mathcal{L}$ defined as a function of the field $\phi,\dot{\phi},\nabla\phi$, the differential action is given by (in terms of the four-field ϕ_a):

$$\partial S=\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)\right]$$
$$=\int d^4x\left\{\left[\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]\delta\phi_a+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)\right\}$$

Euler-Lagrange equation for four-fields (CBG, Pg 8) The minimization of the action for the Lagrangian density (and the requirement that the field vanish at spatial infinity to remove the boundary term) gives:

$$\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)=0$$

which can be written:

$$\frac{\partial\mathcal{L}}{\partial\phi_a}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\phi_a}}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\phi_a)_x}=0$$

where x runs over all x,y,z.

Lagrangian for real scalar field (CBG, Pg 8)

$$\mathcal{L}=\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2$$
$$=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2$$
$$=\frac{1}{2}\dot{\phi}^2-\frac{1}{2}(\nabla\phi)^2-\frac{1}{2}m^2\phi^2$$

Note that the partial with respect to the covariant gradient is:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=\partial^\mu\phi=(\dot{\phi},-\nabla\phi)$$

so that the equation of motion is:

$$\ddot{\phi}-\nabla^2\phi+m^2\phi=0$$
$$\iff\partial_\mu\partial^\mu\phi+m^2\phi=0$$
$$\iff\Box\phi+m^2\phi=0$$

We can also construct the Hamiltonian for the system (P 2.8, Pg 17):

$$\pi(\boldsymbol{x}) = \dot{\phi}(\boldsymbol{x})$$
$$\implies \mathcal{H} = \frac{\pi^2}{2} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$
$$H = \int d^3x\mathcal{H}$$

Maxwell's equation in Lagrangian form (CBG Pg 10)

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2$$

Momentum density conjugate (P 2.4, Pg 16)

$$\pi(\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\boldsymbol{x})}$$

Hamiltonian density (P 2.5, Pg 16)

$$\mathcal{H} = \pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathcal{L}$$
$$H = \int d^3x\mathcal{H}$$

QFT Commutation relations (P 2.20, Pg 20) note $\hbar = 1$.

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})$$
$$[\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = 0$$
$$[\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0$$

Fourier expansion of field (P Pg 20)

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\phi(\boldsymbol{p}, t)$$

where the condition for $\phi(\boldsymbol{x}, t)$ to be real is that:

$$\phi^*(\boldsymbol{p}, t) = \phi(-\boldsymbol{p}, t)$$

Klein-Gordon equation in momentum space (P 20.21, Pg 20)

$$\left[\frac{\partial}{\partial t^2} + (|\boldsymbol{p}|^2 + m^2)\right]\phi(\boldsymbol{p}, t) = 0$$

which has the form of a simple harmonic oscillator with frequency:

$$\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$$

Review of Simple Harmonic Oscillator (P 2.23-24, Pg 20) For the Hamiltonian:

$$H_{SHO} = \frac{p^2}{2} + \frac{1}{2}\omega^2\phi^2$$

the ladder operators are related to the position and momentum operators by:

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$$
$$p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

with canonical commutation relation $[a, a^\dagger] = 1$ so that the Hamiltonian becomes:

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right)$$
$$[H, a^\dagger] = \omega a^\dagger$$
$$[H, a] = -\omega a$$

Klein-Gordon Hamiltonian spectrum (P 2.25-26, Pg 21) Let each Fourier mode of the field be treated as an independent harmonic oscillator:

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)$$
$$\pi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)$$

Equivalently (requiring the integral kernel to satisfy $\phi^\dagger(\boldsymbol{p}) = \phi(-\boldsymbol{p})$ to ensure that the LHS is purely real),

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$
$$\pi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$

and the commutation relations are:

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')$$
$$[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}] = 0$$
$$[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = 0$$
$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = i\delta^3(\boldsymbol{x}-\boldsymbol{x}')$$

The commutation of the raising operators implies that the Klein-Gordon particles obey Bose-Einstein statistics (order of adding particles does not matter). **Interpretation of K-G operators (P 2.41-2.42)** $\phi(\boldsymbol{x})$ creates a particle at position $\boldsymbol{x}$:

$$\phi(\boldsymbol{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}|\boldsymbol{p}\rangle$$
$$\implies \langle 0|\phi(x)|p\rangle = e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$

in analogy to the non-relativistic result $\langle\boldsymbol{x}|\boldsymbol{p}\rangle = e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$.

Field Hamiltonian using ladder operators (P 2.31, Pg 21)

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + \frac{1}{2}[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}]\right)$$

Note that the second term is proportional to $\delta(0)$, or infinity. The commutation relations with the ladder operators are very similar to the SHO case:

$$[H, a^\dagger_{\boldsymbol{p}}] = \omega_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}$$
$$[H, a_{\boldsymbol{p}}] = -\omega_{\boldsymbol{p}}a_{\boldsymbol{p}}$$

Field momentum operator (P 2.33, Pg 22)

$$\boldsymbol{P} = -\int d^3x\pi(\boldsymbol{x})\nabla\phi(\boldsymbol{x})$$
$$= \int \frac{d^3p}{(2\pi)^3}\boldsymbol{p}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}$$

Constructing field states The state $a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}\cdots|0\rangle$ has momentum $\boldsymbol{p}+\boldsymbol{q}+\ldots$ and energy $\omega_{\boldsymbol{p}}+\omega_{\boldsymbol{q}}+\ldots$. **Field vacuum state** satisfies:

$$a_{\boldsymbol{p}}|0\rangle = 0$$
$$\langle 0|0\rangle = 1$$

Delta function of function (P 2.34, Pg 22)

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

More generally, for a function $g(x)$ with roots at x_i,

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

Normalized K-G state (P 2.35, Pg 23) Noting that $\omega_{\boldsymbol{p}} = E_{\boldsymbol{p}}$,

$$|\boldsymbol{p}\rangle = \sqrt{2E_{\boldsymbol{p}}}a^\dagger_{\boldsymbol{p}}|0\rangle$$
$$\langle\boldsymbol{p}|\boldsymbol{q}\rangle = 2E_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})$$

where the normalization is chosen to be Lorentz invariant.

Resolution of identity (P 2.39, Pg 23) Given the above normalization and for one-particle:

$$\mathbb{I} = \int \frac{d^3p}{(2\pi)^3}|\boldsymbol{p}\rangle\frac{1}{2E_{\boldsymbol{p}}}\langle\boldsymbol{p}|$$

Lorentz-invariant 3-momentum integral (P 2.40, Pg 23)

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2)\theta(p^0)$$

Heisenberg Picture Operators gain time dependence:

$$\phi(\boldsymbol{x}, t) = e^{iHt}\phi(\boldsymbol{x}, 0)e^{-iHt}$$
$$i\frac{\partial O}{\partial t} = [O, H]$$

note that H is the full Hamiltonian, that is, the integrated Hamiltonian density. **Time evolution of K-G operators**

in Hamiltonian picture (P 2.45, Pg 25)

$$i\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = i\pi(\boldsymbol{x},t)$$
$$i\frac{\partial\pi(\boldsymbol{x},t)}{\partial t} = -i(-\nabla^2 + m^2)\phi(\boldsymbol{x},t)$$

where we use the commutation relations between ϕ and π, and let ϕ commute with $\nabla\phi$. The raising and lowering operators obey:

$$e^{iHt}a_p e^{-iHt} = a_p e^{-iE_p t}$$
$$e^{iHt}a_p^\dagger e^{-iHt} = a_p^\dagger e^{iE_p t}$$

which can be proven by using the commutation relations to derive:

$$H^n a_p = a_p(H - E_p)^n$$

The K-G operators have time-dependent form:

$$\phi(\boldsymbol{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_p e^{-ipx} + a_p^\dagger e^{ipx})\Big|_{p^0=E_p}$$
$$\pi(\boldsymbol{x},t) = \frac{\partial}{\partial t}\phi(\boldsymbol{x},t)$$

where px is the four-dimensional scalar product:

$$px = p^0 t - \boldsymbol{p}\cdot\boldsymbol{x} = p^\mu \eta_{\mu\nu} x^\mu$$

Displaced position operator (P 2.48-49, Pg 26) Using commutation relations, one can show

$$e^{-i\boldsymbol{P}\cdot\boldsymbol{x}}a_p e^{i\boldsymbol{P}\cdot\boldsymbol{x}} = a_p e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$
$$e^{-i\boldsymbol{P}\cdot\boldsymbol{x}}a_p^\dagger e^{i\boldsymbol{P}\cdot\boldsymbol{x}} = a_p e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}$$

where $\boldsymbol{P}$ is the total system 3-momentum operator, and $\boldsymbol{p}$ is the three-momentum associated with a_p (just one Fourier mode). Then:

$$\phi(x) = e^{iPx}\phi(0)e^{-iPx}$$

where $P^\mu = (H, \boldsymbol{P})$ and Px is a four-vector scalar product.

Name	$W(x)$	$V_1(x;a)$	$f(a)$	$E_n^{(1)}$
Harmonic Oscillator	$\frac{1}{2}ax - b$	$\frac{a^2}{4}(x - \frac{2b}{a})^2 - a/2$	a	na
3D Oscillator	$\frac{1}{2}\omega r - \frac{a+1}{r}$	$\frac{1}{4}\omega^2 r^2 + \frac{a(a+1)}{r^2} - (a+3/2)\omega$	$a+1$	$2n\omega$
Coulomb	$\frac{e^2}{2(a+1)} - \frac{a+1}{r}$	$-\frac{e^2}{r} + \frac{a(a+1)}{r^2} + \frac{e^4}{4(a+1)^2}$	$a+1$	$\frac{e^4}{4(a+1)^2} - \frac{e^4}{4(n+a+1)^2}$