Outline:

1. Consumption: Basic model and early theories

2. Linearization of the Euler Equation

3. Empirical tests without “precautionary savings effects”
1 Application: Consumption.

Sequence Problem (SP): Find $v(x)$ such that

$$v(x_0) = \sup_{\{c_t\}_{t=0}^\infty} E_0 \sum_{t=0}^\infty \delta^t u(c_t)$$

subject to a static budget constraint for consumption,

$$c_t \in \Gamma^C(x_t),$$

and a dynamic budget constraint for assets,

$$x_{t+1} \in \Gamma^X(x_t, c_t, \tilde{R}_{t+1}, \tilde{y}_{t+1}, \ldots).$$

Here $x$ is the vector of assets, $c$ is consumption, $R$ is the vector of financial asset returns, and $y$ is the vector of labor income.
For instance, consider the case where the only asset is cash-on-hand (so $x$ is cash-on-hand) and consumption is constrained to lie between 0 and $x$. Then,

$$c_t \in \Gamma^C(x_t) \equiv [0, x_t]$$

$$x_{t+1} \in \Gamma^X(x_t, c_t, \tilde{R}_{t+1}, \tilde{y}_{t+1}) \equiv \tilde{R}_{t+1}(x_t - c_t) + \tilde{y}_{t+1}$$

$$x_0 = y_0.$$
• We will always assume that $\tilde{y}$ is exogenous and iid.
  
  – However, in the welfare state, $\tilde{y}$ is not independent of $x$. See Hubbard, Skinner, and Zeldes (1995).

• We will always assume that $u$ is concave ($u'' < 0$ for all $c > 0$).

• We will usually assume $\lim_{c \downarrow 0} u'(c) = \infty$, so $c > 0$ as long as $x > 0$.
  
  – I’ll highlight exceptions to this rule.
1.1 Bellman Equation representation

- The state variable, $x$, is stochastic, so it is not directly chosen (rather a distribution for $x_{t+1}$ is chosen at time $t$).

- It is more convenient to think about $c$ as the choice variable.

Bellman Equation:

$$v(x) = \sup_{c \in [0,x]} \left\{ u(c) + \delta Ev(x+1) \right\} \quad \forall x$$

$$x_{t+1} = \tilde{R}_{t+1}(x - c) + \tilde{y}_{t+1}$$

$$x_0 = y_0.$$
1.2 Necessary Conditions

• First Order Condition:

\[ u'(c_t) = \delta E_t \tilde{R}_{t+1} v'(x_{t+1}) \text{ if } 0 < c_t < x_t \]
\[ u'(c_t) \geq \delta E_t \tilde{R}_{t+1} v'(x_{t+1}) \text{ if } c_t = x_t \]

• Envelope Theorem: \( v'(x) = u'(c) \). Prove this. What if \( c = x \)?

• Euler Equation:

\[ u'(c_t) = \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \text{ if } 0 < c_t < x_t \]
\[ u'(c_t) \geq \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \text{ if } c_t = x_t \]
1.3 Perturbation intuition behind the Euler Equation:

- What is the cost of consuming $\varepsilon$ dollars less today?

  Utility loss today = $\varepsilon \cdot u'(c_t)$

- What is the (discounted, expected) gain of consuming $\tilde{R}_{t+1}$ dollars more tomorrow?

  Utility gain tomorrow = $\delta E_t \left[ (\tilde{R}_{t+1} \varepsilon) \cdot u'(c_{t+1}) \right]$

Let’s now rederive the Euler Equation:
1. Suppose \( u'(c_t) < \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \). Then cut \( c_t \) by \( \varepsilon \) and raise \( c_{t+1} \) by \( \tilde{R}_{t+1} \varepsilon \) to generate a net utility gain:

\[
\varepsilon \cdot \left[ -u'(c_t) + \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \right] > 0.
\]

This perturbation is always possible on the equilibrium path, so:

\[
u'(c_t) \geq \delta E_t \tilde{R}_{t+1} u'(c_{t+1}).
\]

2. Suppose \( u'(c_t) > \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \), then raise \( c_t \) by \( \varepsilon \) and cut \( c_{t+1} \) by \( \tilde{R}_{t+1} \varepsilon \) to generate a net utility gain:

\[
\varepsilon \cdot \left[ u'(c_t) - \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \right] > 0.
\]

This perturbation is possible on the equilibrium path as long as \( c < x \), so:

\[
u'(c_t) \leq \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \quad \text{as long as} \quad c < x.
\]
It follows that

\[ u'(c_t) = \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \quad \text{if} \quad c_t < x_t \]
\[ u'(c_t) \geq \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \quad \text{if} \quad c_t = x_t \]
1.4 Important consumption models:

1.4.1 Life Cycle Hypothesis: Modigliani & Brumberg (1954)

- $\tilde{R}_t = R, \delta R = 1$.

- Perfect capital markets (and no moral hazard), so that future labor income can be exchanged for current capital.
Bellman Equation:

\[ v(x) = \sup_{c \leq x} \{ u(c) + \delta Ev(x_{+1}) \} \quad \forall x \]

\[ x_{+1} = R(x - c) \]

\[ x_0 = E \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t \]

- Sometimes referred to as “eating a pie/cake problem.”

- Euler Equation implies,

\[ u'(c_t) = \delta Ru'(c_{t+1}) = u'(c_{t+1}). \]

- Hence, consumption is constant.
Budget constraint:

\[ \sum_{t=0}^{\infty} R^{-t} c_t = E_0 \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t \]

Substitute Euler Equation, \( c_0 = c_t \) to find

\[ \sum_{t=0}^{\infty} R^{-t} c_0 = E_0 \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t \]

So Euler Equation + budget constraint implies

\[ c_0 = \left( 1 - \frac{1}{R} \right) \left( E_0 \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t \right) \quad \forall t \]

Consumption is an annuity. What’s the value of your annuity?

**Remark 1.1** Friedman’s Permanent Income Hypothesis (Friedman, 1957) is much like Modigliani’s Life-Cycle Hypothesis.
1.4.2 Certainty Equivalence Model: Hall (1978)

- $\tilde{R}_t = R, \delta R = 1$

- Quadratic utility: $u(c) = \alpha c - \frac{\beta}{2}c^2$
  
  - This admits negative consumption.

  - And this does not imply that $\lim_{c \downarrow 0} u'(c) = \infty$.

- Can’t sell claims to labor income.
• **Bellman Equation:**

\[
v(x) = \sup_c \{ u(c) + \delta Ev(x+1) \} \quad \forall x
\]

\[
x_{+1} = R(x - c) + \tilde{y}_{+1}
\]

\[
x_0 = y_0
\]

\[
\sum_{t=0}^{\infty} R^{-t} c_t \leq \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t \tag{BC}
\]

• Euler Equation implies, \( c_t = E_t c_{t+1} = E_t c_{t+n} \) (consumption is a random walk w/o drift):

\[
c_{t+1} = c_t + \varepsilon_{t+1}.
\]

• So \( \Delta c_{t+1} \) can not be predicted by any information available at time \( t \).
Budget constraint at date $t$:

$$
\sum_{s=0}^{\infty} R^{-s} c_{t+s} = x_t + \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}
$$

$$
E_t \sum_{s=0}^{\infty} R^{-s} c_{t+s} = x_t + E_t \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}
$$

Substitute $c_t = E_t c_{t+s}$ to find

$$
\sum_{s=0}^{\infty} R^{-s} c_t = x_t + E_t \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}
$$

So Euler Equation + budget constraint implies

$$
c_t = \left(1 - \frac{1}{R}\right) \left(x_t + E_t \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}\right) \quad \forall t
$$
2 Linearizing Euler Equation

Recall Euler Equation:

\[ u'(c_t) = E_t \delta R_{t+1} u'(c_{t+1}) \]

Want to transform this equation so it is more amenable to empirical analysis.

Assume that \( R_{t+1} \) is known at time \( t \).

Assume \( u \) is an isoelastic (i.e., constant relative risk aversion) utility function, \[
 u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}.
\]

(Aside: \( \lim_{\gamma \to 1} \frac{c^{1-\gamma} - 1}{1-\gamma} = \ln c \). Important special case.)
Note that
\[ u'(c) = c^{-\gamma}. \]

We can rewrite the Euler Equation as
\[ c_t^{-\gamma} = E_t \delta R_{t+1} c_{t+1}^{-\gamma} \]
\[ 1 = E_t \exp \left[ \ln \left( \delta R_{t+1} c_{t+1}^{-\gamma} c_t^\gamma \right) \right] \]
\[ 1 = E_t \exp \left[ -\rho + r_{t+1} - \gamma \ln(c_{t+1}/c_t) \right] \]

where \(-\ln \delta = \rho\) and \(\ln R_{t+1} = r_{t+1}\).

Since, \(\ln(c_{t+1}/c_t) = \ln(c_{t+1}) - \ln(c_t)\), we write,
\[ 1 = E_t \exp \left[ r_{t+1} - \rho - \gamma \Delta \ln c_{t+1} \right]. \]
Assume that $\Delta \ln c_{t+1}$ is conditionally normally distributed. So,

$$1 = \exp \left[ E_t r_{t+1} - \rho - \gamma E_t \Delta \ln c_{t+1} + \frac{1}{2} \gamma^2 V_t \Delta \ln c_{t+1} \right].$$

Taking the natural log of both sides yields,

$$E_t \Delta \ln c_{t+1} = \frac{1}{\gamma} (E_t r_{t+1} - \rho) + \frac{\gamma}{2} V_t \Delta \ln c_{t+1}.$$
3 Empirical tests without precautionary savings effects

Recall Euler Equation:

\[ u'(c_t) = E_t \delta R_{t+1} u'(c_{t+1}) \]

We write the linearized Euler Equation in regression form:

\[ \Delta \ln c_{t+1} = \frac{1}{\gamma} (E_t r_{t+1} - \rho) + \frac{\gamma}{2} V_t \Delta \ln c_{t+1} + \varepsilon_{t+1} \]

where \( \varepsilon_{t+1} \) is orthogonal to any information known at date \( t \).

The conditional variance term is often referred to as the “precautionary savings term,” (more on this later).
We sometimes (counterfactually) assume that $V_t \Delta \ln c_{t+1}$ is constant (i.e., independent of time). So the Euler Equation reduces to:

$$\Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} (E_t r_{t+1} - \rho) + \epsilon_{t+1}$$

When we replace the precautionary term with a constant, we are effectively ignoring its effect (since it is no longer separately identified from the other constant term: $\frac{\rho}{\gamma}$).
Hundreds of papers have estimated a linearized Euler Equation:

$$\Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} E_t r_{t+1} + \beta X_t + \varepsilon_{t+1}$$

The principal goals of these regressions are twofold:

1. Estimate $\frac{1}{\gamma}$, the elasticity of intertemporal substitution (EIS) = $\frac{\partial \Delta \ln c_{t+1}}{\partial E_t r_{t+1}}$. For example, see Hall (1988).

   • For this model, the EIS is the inverse of the CRRA.
2. Test the orthogonality restriction: \( \{ \Omega_t \equiv \text{information set at date } t \} \perp \varepsilon_{t+1} \).

- In other words, test the restriction that information available at time \( t \) does not predict consumption growth in the following regression

\[
\Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} E_t r_{t+1} + \beta X_t + \varepsilon_{t+1}.
\]

- For example, does the date \( t \) expectation of income growth, \( E_t \Delta \ln Y_{t+1} \), predict date \( t + 1 \) consumption growth?
\[ \Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} E_t r_{t+1} + \alpha E_t \Delta \ln Y_{t+1} + \varepsilon_{t+1} \]

\[ \alpha \in [0.1, 0.8], \text{ so } E_t \Delta \ln Y_{t+1} \text{ covaries with } \Delta \ln c_{t+1} \text{ (e.g., Campbell and Mankiw 1989, Shea 1995, Shapiro 2005, Parker and Broda 2014).} \]

- Orthogonality restriction is violated: information at date \( t \) predicts consumption growth from \( t \) to \( t + 1 \).

- In other words, the assumptions (1) the Euler Equation is true, (2) the utility function is in the CRRA class, (3) the linearization is accurate, and (4) \( V_t \Delta \ln c_{t+1} \) is constant, are jointly rejected.
A note on Shea’s methodology (for estimating $E_t \Delta \ln Y_{t+1}$)

1. Assign respondents to unions with national or regional bargaining

   • national: trucking, postal service, railroads

   • regional: lumber in Pacific Northwest, shipping on East Coast

2. Assign respondents to dominant local employer

   • automobile worker living in Genesee County, MI (GM’s Flint plant)
Why does expected income growth predict consumption growth?

- Leisure and consumption expenditure are substitutes (Heckman 1974, Ghez and Becker 1975, Aguiar and Hurst 2005, 2007)

- Work-based expenses

- Households support lots of dependents in mid-life when income is highest (Browning 1992, Attanasio 1995, Seshadri et al 2006)

• Some consumers use rules of thumb: \( c_{it} = \alpha Y_{it} \) (Campbell and Mankiw 1989, Thaler and Shefrin 1981)

• Welfare costs of smoothing are second-order (Cochrane 1989, Pischke 1995, Browning and Crossley 2001)