Economics 2010c: Lecture 5
Non-stationary Dynamic Programming

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Outline:

1. Non-stationary dynamic programming

2. Lifecycle problem with liquidity constraints

3. Simulated Euler equation tests with liquidity constrained households
1 Non-stationary dynamic programming

- So far we have assumed that problem is stationary.

- I.e., value function does not depend on time, only on the value of the state variable.

- But, the dynamic programming approach is also applicable to non-stationary problems, including “finite horizon” problems.

- E.g., Born at $t = 1$ and terminate at $t = T$ (termination may or may not be the end of “life”).
Two changes arise in finite horizon dynamic programming (i.e. backwards induction).

1. We will now index the value function, since each value function is *date* specific.

   - $v_t(x)$ represents value function for period $t$
   - E.g., $v_t(x) = E_t \sum_{s=t}^T \delta^{s-t} u(c_s)$

2. We will *not* iterate the functional operator on an arbitrary starting function $w$.

   - instead iterate Bellman operator on $v_T(x)$, where $T$ is the last period in the problem or a period at which some “termination payoff” applies.
In most cases, $v_T(x_T)$, the termination payoff, will be easy to calculate.

E.g., suppose the last period is the “end of life,” and there is no bequest motive.

\[ v_T(x) = u(x) \]

What would you do if there were a bequest motive?
A simple recursion links \( v_{t-1}(x) \) and \( v_t(x) \). E.g., consider Bellman Equation,

\[
v_{t-1}(x) = \sup_{c \in [0,x]} \{ u(c) + \delta E v_t(R(x - c) + \tilde{y}) \} \quad \forall x
\]

To generate \( v_{t-1}(x) \) apply Bellman Operator

\[
(Bf)(x) \equiv \sup_{c \in [0,x]} \{ u(c) + \delta E f(R(x - c) + \tilde{y}) \} \quad \forall x
\]

\[
v_{t-1}(x) = (Bv_t)(x).
\]

Generally, we have:

\[
v_{T-n}(x) = (B^n v_T)(x).
\]

Note that Bellman Operator itself could be time-contingent, though that is not the case in this example.
2 Lifecycle problem with liquidity constraints

- iid labor income

\[ y_t = \begin{cases} 
0 & \text{with probability } 1/2 \\
1 & \text{with probability } 1/2 
\end{cases} \]

- \( c_t \leq x_t = \text{cash-on-hand} \)

- \( C R R A = 1 \) (log utility), \( \delta = 0.9, R = 1.03 \)

- 40 periods of working life followed by “infinite” retirement

- In retirement consumer lives off accumulated wealth
\[ v_1(x_1) = \max E_1 \sum_{s=0}^{\infty} \delta^s u(c_{s+1}) \]

\[ x_t = R(x_{t-1} - c_{t-1}) + \tilde{y}_t \quad t = 2, \ldots, 40 \]

\[ x_t = R(x_{t-1} - c_{t-1}) \quad t = 41, \ldots, \infty \]

- Starting at \( t = 40 \), consumer solves eat-the-pie problem with wealth \( x_{40} \).

- \( v_{40}(x) \) solves \( \infty \)-horizon eat-the-pie-problem (coefficients derived in previous lecture):
  \[ v_{40}(x) = \phi + \psi \ln x \]

- For \( n = 1, \ldots, 39 \), \( v_{40-n}(x) = (B^n v_{40})(x) \).
See Figures:

- Consumption functions during working life
- Lifecycle (working life) simulation for a single household
- Cash-on-hand over the lifecycle (working life)
- Consumption over the lifecycle (working life)
Consumption Functions Over Working Life: Periods 1-40
Simulated lifecycle outcomes for one household: cash-on-hand, income, consumption
Average cash-on-hand for 100 households

Building up buffer stock

Buffer stock saving in steady state

Average income is 1/2 and average stock of savings is about 1.

Saving for retirement

Model Age
Average consumption for 100 households
3 Euler equation tests using simulated data

Generate simulated data from 5000 preretirement households. Use consumption functions, \( \{c_t(x_t)\}_{t=1}^{40} \), and the dynamic budget constraint,

\[
x_{t+1} = R(x_t - c_t) + \tilde{y}_{t+1}.
\]

Estimate linearized Euler Equation regression, using simulated panel data.

\[
\Delta \ln c_{t+1} = \text{constant} + X_t \beta + \varepsilon_{t+1}
\]

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<tbody>
<tr>
<td>constant</td>
<td>-0.08</td>
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<td>-0.07</td>
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<td>-0.01</td>
<td>-0.04</td>
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<td>-0.05</td>
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<tr>
<td>( V_t \Delta \ln c_{t+1} )</td>
<td>0.48</td>
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<td>( x_t )</td>
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<td>( y_t )</td>
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<td>( age/10 )</td>
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<td>( E_t \Delta y_{t+1} )</td>
<td>0.00</td>
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The table above provides the coefficients and standard errors for the estimated Euler equation. The parameters are estimated using simulated panel data.
Regressions 1-4 are correctly specified, since they contain the conditional variance term. These regressions match theoretical predictions:

- prediction for constant:
  \[
  \frac{1}{\gamma} (\ln R + \ln \delta) = \frac{1}{1} (\ln 1.03 + \ln 0.9) = -0.076
  \]

- prediction for coefficient on \( V_t \Delta \ln c_{t+1} \):
  \[
  \frac{\gamma}{2} = 0.5
  \]

- prediction for coefficients on other RHS variables are exactly zero, since they represent information available at time \( t \).
Regressions 5-8 omit $V_t \Delta \ln c_{t+1}$, generating omitted variables bias

- variables $x_t$, $y_t$, and age correlate negatively with $V_t \Delta \ln c_{t+1}$, so the estimated coefficients on these variables are negative

- variable $E_t \Delta y_{t+1}$ correlates positively with $V_t \Delta \ln c_{t+1}$, so the estimated coefficients on $E_t \Delta y_{t+1}$ is positive