Economics 2010c: $q$-theory

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Outline:

1. Why should we study investment?

2. Static model

3. Dynamic model: $q$-theory of investment

4. Phase diagrams

5. Analytic example of $q$ Model (optional)
1 Why should we study investment?

- \( K_{t+1} = K_t - \delta K_t + I_{t+1} \) (and \( K \) supports future consumption)

- \( I \) is a volatile component of GDP, so it plays a central role in short-run (i.e., business cycle) fluctuations

- \( I \) represents 15% of GDP

- Dynamics in \( I \) provide insight into investor rationality (e.g., animal spirits) and market institutions.

- Some policies are designed to influence \( I \) (e.g., investment tax credit)
2 Static model

- Firm Profit = $F(k, x_1, x_2, \ldots, x_n) - rk$ where $F_k > 0$, $F_{kk} < 0$

- $r$ is the user cost of capital (Hall and Jorgenson) and $k$ is capital

- First-order-condition: $F_k(k, x_1, \ldots, x_n) = r$

- Differentiate first-order-condition to find:
  \[
  \frac{dk(r, x_1, \ldots, x_n)}{dr} = \frac{1}{F_{kk}(k, x_1, \ldots, x_n)} < 0
  \]

- So capital falls when the interest rate rises.
What’s wrong with this simple model?

- Every period $k$ jumps to new static optimum (instantaneous adjustment)

- Expectations play no role in dynamics of $k$ (e.g., future profitability of capital doesn’t lead to more capital formation today)

- Capital is perfectly divisible (e.g., in the model you can buy half an airplane)
3 q-theory of investment

- Continuum of identical firms (on unit interval)

- $k(t) =$ firm level capital stock

- $K(t) = k(t)$ aggregate capital stock

- Individual firms have no effect on the aggregate capital stock
• \( \pi(K(t)) \) = revenue per unit of aggregate capital

  – in other words, \( \pi \) is total revenue in the industry divided by total capital in the industry

  – downward sloping *industry* demand curve implies that

    \[ \pi'(K(t)) < 0 \]

• \( \pi(K(t)) \times k(t) \) = revenue for a firm with firm-level capital of \( k \)

  – constant returns to scale within the firm (double \( k \) and you double firm revenue)
• $dk = I \ dt$ (no depreciation and no uncertainty)

• $dK = I^{AGG} dt$

• Adjustment costs (the cost of installing or removing capital):
  - $C(I) =$ cost of installing investment $I$; $C(0) = 0$
  - $C''(0) = 0$
  - $C'''(I) > 0$ (convex costs)

• Adjustment costs arise even when $I < 0$. Why?

• Total cost of investment: $I + C(I)$. 
• $V(k, K)$ represents value function of firm

• Instantaneous profit flow: $w(k, K, I) = \pi(K)k - I - C(I)$.

• Continuous time Bellman Equation:

$$V(k, K) = \max_I \left\{ w(k, K, I)\Delta t + (1 + r\Delta t)^{-1}V(k', K') \right\}$$

$$\phantom{V(k, K)} (1 + r\Delta t)V(k, K) = \max_I \left\{ (1 + r\Delta t)w(k, K, I)\Delta t + V(k', K') \right\}$$

Multiply out and let $\Delta t \to 0$. Set terms of order $(dt)^2$ equal to 0.

$$rV(k, K)dt = \max_I \left\{ w(k, K, I)dt + dV \right\}$$

$$= \max_I \left\{ [\pi(K)k - I - C(I)]dt + dV \right\}$$
• \( q(t) \) = value of marginal unit of **installed** capital

\[
q(t) = \int_{\tau=t}^{\infty} e^{-r(\tau-t)} \pi(K(\tau)) d\tau.
\]

• We can also express \( q(t) \) in terms of a Bellman Equation:

\[
q(t) = \pi(K(t)) \Delta t + (1 + r \Delta t)^{-1} q(t')
\]

\[
r q(t) dt = \pi(K(t)) dt + dq
\]

• Since \( q \) is not an Ito Process, \( \frac{dq}{dt} = \dot{q} \) is well-defined:

\[
r q = \pi(K(t)) + \dot{q}
\]
• Since $K$ and $k$ depend only on deterministic terms,

$$dV = \frac{\partial V}{\partial k}Idt + \frac{\partial V}{\partial K}I^{AGG}dt$$

• $\frac{\partial V}{\partial k} = q$ (from intuition or Envelope Theorem)
\[ rV(k, K)dt = \max_I \{[\pi(K)k - I - C(I)]dt + dV\} \]
\[ = \max_I \left\{[\pi(K)k - I - C(I)]dt + \frac{\partial V}{\partial k}I dt + \frac{\partial V}{\partial K}I^{AGG} dt \right\} \]
\[ = \max_I \left\{[\pi(K)k - I - C'(I)]dt + qI dt + \frac{\partial V}{\partial K}I^{AGG} dt \right\} \]

First-order-condition:
\[-1 - C'(I) + q = 0\]

marginal cost = \(1 + C'(I) = q\) = marginal benefit

Note that \(q\) is the sufficient statistic for investment decision.
Dynamics for $K$ and $q$.

First, consider equation for $\dot{K} = I$.

$$1 + C'(I) = q$$

$$I = (C')^{-1}(q - 1)$$

$$\dot{K} = (C')^{-1}(q - 1)$$

So, $\dot{K} = 0$ iff $q = 1$.

Now consider equation for $\dot{q}$.

$$rq = \pi(K) + \dot{q}$$

$$\dot{q} = rq - \pi(K)$$

So, $\dot{q} = 0$ iff $rq = \pi(K)$. 
4 Phase diagrams

\[ \dot{K} = (C')^{-1}(q - 1) \]
\[ \dot{q} = rq - \pi(K) \]

- Steady state: \( q = 1 \) and \( r = \pi(K) \).

- Jump variable: \( q \) (like a derivative price). Can only jump when a surprise occurs. Arbitrage rules out anticipated jumps.

- State variable: \( K \) (a stock). Must always follow smooth dynamics: \( \dot{K} = I \). Investment can’t be infinite.
Experiments:

<table>
<thead>
<tr>
<th>Event Type</th>
<th>( \pi(\cdot) \uparrow )</th>
<th>( r \uparrow )</th>
<th>ITC</th>
</tr>
</thead>
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<tr>
<td>Unanticipated, permanent</td>
<td>✔</td>
<td></td>
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<tr>
<td>Unanticipated, temporary</td>
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ITC represents Investment Tax Credit: every dollar of investment (disinvestment) generates an immediate tax rebate (tax bill) of \( \theta \) dollars.

You should be able to work out the dynamic consequences of these experiments.

We will consider the checked cells in class if time permits.

You should check the other cells on your own (see problem set).
5 Analytic example of $q$ Model (optional)

- $k(t) =$ firm level capital stock
- $K(t) =$ aggregate capital stock (with many small firms)
- profits per firm: $\pi(K(t)) \times k(t)$, where $\pi'(K(t)) < 0$.
- We’ll assume a particular functional form:
  \[
  \pi(K(t)) = 1 - K(t).
  \]
• $dk = I dt$ (no depreciation and no uncertainty)

• $dK = I^{AGG} dt$

• (adjustment) cost of installing investment $I$: $C(I)$.

• We’ll assume a particular functional form:

$$C(I) = \frac{I^2}{2}.$$
\( V(k, K) \) represents value function of firm. From the previous lecture we have:

\[
rv(k, K)dt = \max_I \left\{ [\pi(K)k - I - C(I)]dt + dV \right\}.
\]

\[
\frac{\partial V}{\partial k} = q(t) = \int_t^\infty e^{-r(\tau-t)}\pi(K(\tau))d\tau.
\]

\[
rv = \pi(K(t)) + \dot{q}.
\]

\[
dV = \frac{\partial V}{\partial k}Idt + \frac{\partial V}{\partial K}I^{AGG}dt.
\]

\[
rv(k, K)dt = \max_I \left\{ [\pi(K)k - I - C(I)]dt + qIdt + \frac{\partial V}{\partial K}I^{AGG}dt \right\}.
\]

First-order-condition:

\[
\text{marginal cost} = 1 + C'(I) = q = \text{marginal benefit}.
\]
Our solution must satisfy the following system:

\[ 1 + C'(I) = q \]

\[ rq = \pi(K(t)) + \dot{q}. \]

Plugging in our functional form assumptions yields

\[ 1 + \dot{K} = q \]

\[ rq = 1 - K + \dot{q}. \]

(I have set \( N = 1 \) to simplify notation.)
You can confirm that this system of first-order differential equations has the following solution:

\[ K(t) = \frac{Q}{\theta} + \left(K_0 - \frac{Q}{\theta}\right)e^{-\theta t} \]
\[ q(t) = 1 + (Q - \theta K_0)e^{-\theta t} \]

where

\[ \theta = \frac{-r + \sqrt{r^2 + 4}}{2} \]
\[ Q = \theta(1 - r). \]

(Why do we reject the negative root of \( \theta \)?)

These equations imply a key result:

\[ I = q - 1 = Q - \theta K. \]
Conjectured guess for the value function:

\[ V = Ak + BKk + CK + DK^2 + E. \]

Bellman Equation.

\[ rV(k, K) dt = \max_I \left\{ \left[ \pi(K)k - I - C(I) \right] dt + qI dt + \frac{\partial V}{\partial K} I^{AGG} dt \right\} \]

\[ rAk + rBKk + rCK + rDK^2 + rE = \]

\[ (1 - K)k - (Q - \theta K) - \frac{1}{2}(Q - \theta K)^2 + \]

\[ (Q - \theta K + 1)(Q - \theta K) + (Bk + C + 2DK) (Q - \theta K) \]
Combining terms we get:

\[
\begin{align*}
   rA &= 1 + BQ \\
   rB &= -1 - B\theta \\
   rC &= -Q\theta - C\theta + 2DQ \\
   rD &= \frac{\theta^2}{2} - 2D\theta \\
   rE &= \frac{Q^2}{2} + CQ
\end{align*}
\]

It’s an algebraic mess, but we can confirm our conjecture, by solving these five equations for our five unknowns.
We have shown that the Bellman equation takes the following form:

\[ V = Ak + BKk + CK + DK^2 + E. \]

We can calculate the marginal value of a unit of installed capital:

\[ A + BK = q = Q - \theta K + 1. \]

And compare this to the average value of a unit of installed capital:

\[ \frac{Ak + BKk + CK + DK^2 + E}{k} = A + BK + \frac{CK + DK^2 + E}{k}. \]

Why is the marginal value of a unit of installed capital different from the average value of a unit of installed capital? Hint: think about the value of being allowed to invest? (This model assumes a fixed number, \( N \), of suppliers.)