Economics 2010c: Lecture 12
Discrete Adjustment

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Outline:

1. Empirical evidence on investment

2. Lumpy investment introduction

3. Lumpy investment models: Bertola and Caballero (1990)

4. Lumpy investment and delayed responses

5. Lumpy decisions across economics

6. Optional: Ergodic distributions and the Kolmogorov Equation
1 Empirical evidence on investment

- User cost of capital doesn’t matter at high frequency (Shapiro 1985).

- Does user cost matter at low frequency? (Caballero 1999).

- User cost matters more when powerful instruments are used, like tax code changes (Goolsbee 1998; Cummins, Hassett and Hubbard 1994, Zwick 2013).

- Uncertainty shocks may play a role in driving investment dynamics (Bloom, 2009)
• Cash flow matters more than it should (Fazzari, Hubbard, Petersen 1988, 2000; Lamont 1997, Kaplan and Zingales 1997)

\[
\frac{I}{k} = \alpha \frac{\text{cash-flow}}{k} + \beta q + \text{controls}
\]

• Cash flow matters the most in companies with overconfident CEO’s (Malmendier and Tate 2005).

• But, if \( q \) is well-measured, cash-flow matters less (Cummins, Hassett, & Oliner 2002).

• Open debate about the rationality of real investment (e.g., Zwick 2013).
2 Lumpy investment introduction

- Plant level data suggests that individual establishments do NOT smooth adjustment of their capital stocks

- Instead, adjustment is lumpy, often taking place very quickly

- E.g., firms build a new plant all at once, not a little bit each year
Doms and Dunne (1993): panel of 12,000 U.S. manufacturing plants (1972-89)

- more than half of the establishments exhibited capital growth close to 50% in a single year

- measure of investment concentration
  \[
  \frac{\text{largest investment episode for establishment } i}{\text{total sample investment for establishment } i}
  \]

- if investment were evenly spread this ratio would be \( \frac{1}{18} = 0.06 \)

- instead average value of the ratio is 0.25

- i.e., on average plants did 25% of their 18 year investment in a single year
3 Lumpy investment model: Bertola and Caballero (1990)

- To adjust capital stock, pay fixed cost and variable cost

- When investment occurs, capital stock jumps *instantaneously* by $I$

- This is *discrete* adjustment ($I$ is now *not* a flow but a change in stock)

- We’ll assume that cost function is affine
Assume direction-specific costs (Up and Down)

- fixed costs: $C_D$ and $C_U$

- variable costs include adjustment costs and the cost of purchasing (or gain from selling) capital
  - when $I$ is negative, then variable costs are $c_D|I|$
  - so the slope of the cost function is $-c_D$ when $I$ is negative
  - when $I$ is positive, then variable costs are $c_U I$
- in almost all cases $c_U > 0$, since capital is costly and installation is costly.

- sign of $c_D$ is ambiguous.

- if deinstallation is highly costly, then $c_D > 0$.

- if deinstallation is relatively inexpensive (and second-hand capital has value), then $c_D < 0$.

- our analysis is completely general (but i’ll plot the case $c_D > 0$).
Contrast smooth convex cost function with affine cost function:

- with smooth convex cost function small adjustments are costlessly reversible

- with affine cost function small adjustments are costly to reverse

- this is true even when $C_D = C_U = 0$ (as long $c_U + c_D > 0$)

- so with lumpy cost function it is optimal to wait to respond to small shocks (since you might want to reverse that adjustment later)
Firm’s problem with affine cost functions

- Firm loses profits if the actual capital stock $x$ deviates from a fixed target capital stock $x^*$

- deviations, $X = (x - x^*)$, generate instantaneous negative payoff:
  \[ -\frac{b}{2}X^2 = -\frac{b}{2}(x - x^*)^2 \]

- $X$ is an Ito Process between adjustments
  \[ dX = \alpha dt + \sigma dz \]
  where $dz$ are Brownian increments

- During adjustment $x$ jumps to $x + I$ and $X$ jumps to $X + I$
\[ V(X) = \max E \left\{ \int_{\tau_0}^{\infty} e^{-\rho(\tau-t)} \left( -\frac{b}{2} X^2 \right) d\tau - \sum_{n=1}^{\infty} e^{-\rho(\tau(n)-t)} A(n) \right\} \]

where

\[ A(n) = \begin{cases} 
C_U + c_U I_n & \text{if } I_n > 0 \\
C_D + c_D |I_n| & \text{if } I_n < 0 
\end{cases} \]

- \( \tau(n) \): date of the \( n'\text{th} \) adjustment
- \( A(n) \): cost of the \( n'\text{th} \) adjustment
- \( I_n \): investment in \( n'\text{th} \) adjustment
Ito’s Lemma:

\[ E(dV) = \left[ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial X} \alpha + \frac{1}{2} \frac{\partial^2 V}{\partial X^2} \sigma^2 \right] dt \]

Bellman Equation between adjustments:

\[ \rho V(X) = -\frac{b}{2} X^2 + \alpha V'(X) + \frac{1}{2} \sigma^2 V''(X) \]

Let \( X = U \) represent the point at which capital stock is adjusted up.

Let \( X = u \) represent the point to which the capital stock is raised.

Let \( X = D \) represent the point at which capital stock is adjusted down.

Let \( X = d \) represent the point to which the capital stock is lowered.
In the “action regions”:

If $X \leq U$

$$V(X) = V(u) - [C_U + c_U(u - X)]$$

$$V'(X) = c_U$$

If $X \geq D$

$$V(X) = V(d) - [C_D + c_D(X - d)]$$

$$V'(X) = -c_D$$
Boundary conditions at $U, u, d, D$:

**Value Matching:**
\[
\lim_{X \uparrow U} V(X) = V(u) - [C_U + c_U(u - U)] = V(U) = \lim_{X \downarrow U} V(X)
\]
\[
\lim_{X \downarrow D} V(X) = V(d) - [C_D + c_D(D - d)] = V(D) = \lim_{X \uparrow D} V(X)
\]

**Smooth Pasting:**
\[
\lim_{X \uparrow U} V'(X) = c_U = V'(U) = \lim_{X \downarrow U} V(X)
\]
\[
\lim_{X \downarrow D} V'(X) = -c_D = V'(D) = \lim_{X \uparrow D} V(X)
\]
First order conditions for optimal investment (marginal benefit = marginal cost):

\[ V'(u) = c_U \]
\[ -V'(d) = c_D \]

Final Problem Set: Find a function \( V \) and parameters \( U, u, d, D \) that satisfy the Bellman Equation and the six boundary conditions.
3.1 Comparative Statics

- \( c_U = c_D = 0 \) (no variable cost) \( \implies u = d \)

- \( C_U = C_D = 0 \) (no fixed cost) \( \implies U = u \) and \( D = d \)

- depreciation \( \uparrow \) (i.e., \( \alpha \downarrow \)) \( \implies U, u, d, D \uparrow \)
  - can even find \( U < 0 < u < d < D \) (if \( C_U > 0 \)).

- \( \sigma \uparrow \implies U, u \downarrow \) and \( d, D \uparrow \)

- \( b \uparrow \implies U \uparrow \) and \( D \downarrow \) and \( u, d \to 0 \).
4 Lumpy investment and delayed responses

- Lumpy adjustment at firm level generates smooth adjustment at macro level.

- Need idiosyncratic (firm-level) shocks
  - (plus standard aggregate shocks)
4.1 Macroeconomic Dynamics

Consider an aggregate shock that hits a system in steady state (i.e. a system with the ergodic density):

- some firms adjust immediately
- an unusually high fraction of firms are poised to adjust in the next period
- slowly system returns to ergodic distribution
- hence aggregate response to an aggregate shock is “smooth”
Sometimes system is out of steady state (if there have been recent aggregate shocks).

- System is frequently out of steady state if aggregate shocks are large relative to idiosyncratic (firm-level) shocks.

- In this case, aggregate adjustment will be more like individual adjustment — lumpy

- Simulations must track firm-level distribution of $X$ values
Bottom line:

- short-run responsiveness of economy depends on the fraction of firms that are near the adjustment boundary

- when capital depreciates more quickly...
  - ergodic density is closer to uniform
  - there are more firms near the boundary
  - shocks that encourage capital formation have a bigger short-run impact

- when past aggregate shocks have moved firms closer to the boundary, additional aggregate shocks in the same direction will have a bigger short-run impact (interaction effect)
5 Lumpy decisions across economics

- Inventories, Money demand (Baumol-Tobin inventory model)
- Price setting (Caplin and Spulber 1987, Golosov and Lucas 2005)
- Portfolio adjustment, including housing and durables (Grossman and Laroque 1990)
- Labor hiring/firing (Bentolila and Bertola 1990, Caballero, Engel and Haltiwanger 1997)
- Investment (Dixit 1989, Caballero and Engel 1999, Bloom 2009)
Figure 1: Monthly US stock market volatility

Notes: CBOE VXO index of % implied volatility, on a hypothetical at the money S&P100 option 30 days to expiration, from 1986 to 2007. Pre 1986 the VXO index is unavailable, so actual monthly returns volatilities calculated as the monthly standard-deviation of the daily S&P500 index normalized to the same mean and variance as the VXO index when they overlap (1986-2006). Actual and VXO are correlated at 0.874 over this period. The market was closed for 4 days after 9/11, with implied volatility levels for these 4 days interpolated using the European VX1 index, generating an average volatility of 58.2 for 9/11 until 9/14 inclusive. A brief description of the nature and exact timing of every shock is contained in Appendix A. Shocks defined as events 1.65 standard deviations about the Hodrick-Prescott detrended ($\lambda=129,600$) mean, with 1.65 chosen as the 5% significance level for a one-tailed test treating each month as an independent observation.

* For scaling purposes the monthly VXO was capped at 50 for the Black Monday month. Un-capped value for the Black Monday month is 58.2.)
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Half-semester outline:

1. Discrete-time dynamic programming
   - Bellman Equation
   - Contraction Mapping Theorem
   - Blackwell’s Theorem
   - Applications: Growth, Search, Consumption, Asset Pricing
2. Continuous-time dynamic programming

- Brownian Motion, Ito Processes, Ito’s Lemma
- Bellman Equation
- Boundary conditions: Value matching and smooth pasting
- Solving second-order differential equations
- Applications: Asset pricing, stopping problems, investment
6 Ergodic Distribution (optional)

Ergodic (steady state) density: $f(X)$.

Two interpretations:

1. If you observe a ‘large’ panel of firms with idiosyncratic shocks, the distribution of firms in $X$ space will converge to $f(X)$, whatever the initial starting points for the individual firms.

2. If you observe a single firm at a point in time, that firm’s ‘unconditional’ location density in $X$ space will be given by $f(X)$. 
Recall the discrete time approximation for an Ito Process (see special notes on stochastic calculus and Brownian Motion): 

\[ \Delta X = \begin{cases} 
-h \text{ with probability } \frac{1}{2}(1 - \frac{a}{\sigma}\sqrt{\Delta t}) \\
+h \text{ with probability } \frac{1}{2}(1 + \frac{a}{\sigma}\sqrt{\Delta t}) 
\end{cases} \]

where

\[ h = \sigma(\Delta t)^{\frac{1}{2}} \]
We characterize $f(\cdot)$ and its associated (Kolmogorov) balance equation:

$$f(X) = f(X + h) \frac{1}{2} (1 - \frac{a}{\sigma} \sqrt{\Delta t}) + f(X - h) \frac{1}{2} (1 + \frac{a}{\sigma} \sqrt{\Delta t}).$$

Rearranging yields,

$$0 = \frac{1}{2} \{[f(X + h) - f(X)] - [f(X) - f(X - h)]\} - \frac{a}{\sigma} \frac{\sqrt{\Delta t}}{2} \{[f(X + h) - f(X)] + [f(X) - f(X - h)]\}.$$

Dividing through by $h^2$ and then letting $\Delta t \to 0$ yields,

$$0 = f''(X) - \frac{2a}{\sigma^2} f'(X).$$

If $a \neq 0$, then the general solution to this equation is

$$f(X) = Ae^{rX} + B \quad \quad r = \frac{2a}{\sigma^2}.$$
• Note that the characterization on the previous slide analyzed values of $X$ that can be approached only from either $X + h$ and $X - h$. Almost all points in $X$ have this property.

• But four special values of $X$ do not have this general property: $U, u, d, D$.

• By construction, $U$ can never be realized, since you jump to $u$ instead of realizing $U$.

• Likewise, $D$ can never be realized.

• By contrast, $u$ and $d$ can be approached from either the right or the left OR the respective boundaries $U$ and $D$. 
We can analyze the special balance equation at $u$.

$$f(u) = f(u-h) \frac{1}{2}(1+\frac{a}{\sigma}\sqrt{\Delta t}) + f(u+h) \frac{1}{2}(1-\frac{a}{\sigma}\sqrt{\Delta t}) + f(U+h) \frac{1}{2}(1-\frac{a}{\sigma}\sqrt{\Delta t})$$

Noting that $f(U) = 0$ and rearranging previous equation yields

$$f(u) - f(u - h) = f(u + h) - f(u) + f(U + h) - f(U) - \frac{a}{\sigma}\sqrt{\Delta t} [f(u) - f(u - h)] - \frac{a}{\sigma}\sqrt{\Delta t} [f(u + h) - f(u)] + f(U + h)(-\frac{a}{\sigma}\sqrt{\Delta t})$$

Dividing through by $h$ and letting $\Delta t \to 0$ yields,

$$f'(u^-) = f'(u^+) + f'(U^+).$$

Using analogous reasoning, we can also derive

$$f'(d^+) = f'(d^-) + f'(D^+).$$

Density is not differentiable at the boundaries $(U, u, d, D)$. 
We have three regions within which \( f(X) \) is twice differentiable:

\[(U, u)\]
\[(u, d)\]
\[(d, D)\].

So we have three differential equations and six unknowns:

\[
f(X) = A_1 e^{rX} + B_1
\]
\[
f(X) = A_2 e^{rX} + B_2
\]
\[
f(X) = A_3 e^{rX} + B_3
\]
We have seven boundary conditions, one of which is redundant, leaving us with six effective constraints.

\begin{align}
    f(U) &= 0 \\
    f(u^-) &= f(u^+) \\
    f'(u^-) &= f'(u^+) + f'(U^+) \\
    f'(d^+) &= f'(d^-) + f'(D^+) \\
    f(d^-) &= f(d^+) \\
    f(D) &= 0 \\
    \int_U^L f(X) &= 1
\end{align}
Let’s solve this system.

\[ f(U) = A_1 e^{rU} + B_1 = 0 \]
\[ f(u^-) = A_1 e^{ru} + B_1 = A_2 e^{ru} + B_2 = f(u^+) \]
\[ f'(u^-) = r A_1 e^{ru} = r A_2 e^{ru} + r A_1 e^{rU} = f'(u^+) + f'(U+) \]
\[ f'(d^+) = r A_3 e^{rd} = r A_2 e^{rd} + r A_3 e^{rD} = f'(d^-) + f'(D+) \]
\[ f(d^-) = A_2 e^{rd} + B_2 = A_3 e^{rd} + B_3 = f(d^+) \]
\[ f(D) = A_3 e^{rD} + B_3 = 0 \quad \text{(confirm redundancy)} \]

Finally, the integral restriction implies:

\[
\int_{U}^{L} f(X) = 1 = \int_{U}^{u} \left( A_1 e^{rX} + B_1 \right) dX \\
+ \int_{u}^{d} \left( A_2 e^{rX} + B_2 \right) dX \\
+ \int_{d}^{D} \left( A_3 e^{rX} + B_3 \right) dX
\]
Using the first six equations, we can write all of our variables in terms of $A_1$.

$$B_1 = -A_1 e^{rU}$$
$$A_2 = A_1 \left[1 - e^{r(U-u)}\right]$$
$$B_2 = 0$$
$$A_3 = A_1 \left[1 - e^{r(U-u)}\right] \left[1 - e^{r(D-d)}\right]^{-1}$$
$$B_3 = -A_1 \left[e^{-rD} - e^{-rd}\right]^{-1} \left[1 - e^{r(U-u)}\right]$$

Now we can exploit the integral condition. After a half-dozen pages of simplification, you’ll find,

$$A_1^{-1} = e^{rU} (U - u) + e^{rD} (d - D) \left[1 - e^{r(U-u)}\right] \left[1 - e^{r(D-d)}\right]^{-1}.$$


6.1 Properties of densities:

- when there is no drift in $X$, density is piecewiese linear.

- when there is drift, density is piecewise exponential (fin-shaped; with leading edge facing the direction of the drift)

- depreciation $\uparrow$ (i.e., $\alpha \downarrow$) $\implies f(X) \rightarrow \text{uniform}[U, u]$

- numerical simulations reveal that the expected value of the random variable $X$ is nearly unaffected by variation in the underlying parameters