Economics 2010c: Lectures 9-10
Bellman Equation in Continuous Time

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Outline Lectures 9-10:

1 Continuous-time Bellman Equation

Let's write out the most general version of our problem. Begin with equation of motion of the state variable:

$$dx = a(x, u, t)dt + b(x, u, t)dz$$

Note that dx depends on choice of control u.

Using Ito's Lemma, derive continuous time Bellman Equation:

$$\rho V(x, t) = w(x, u^*, t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} a(x, u^*, t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} b(x, u^*, t)^2$$

$$u^* = u(x, t) = \text{ optimal value of control variable}$$

- Note that value function is a second order *partial differential equation* (PDE).

- V is the 'dependent' variable and x and t are the 'independent' variables.

- To solve this PDE need 'boundary conditions,' since many solutions exist.

1.1 Terminal condition.

- Suppose problem 'ends' at date T

- Then we know that

$$V(x,T) = \Omega(x,T) \quad \forall x$$

- Can solve the problem using techniques analogous to backwards induction

1.2 Stationary $\infty-$horizon problem.

- Now value function doesn't depend on t

$$\rho V(x) = w(x, u^*) + a(x, u^*)V' + \frac{1}{2}b(x, u^*)^2 V''.$$

- We have a second-order *ordinary differential equation* (ODE).

- In general a large class of functions are consistent with this ODE.

- To pin down a solution we need to know something about the economics of the value function V.

- This will provide constraints that pick out a single solution.

2 Application: Merton's consumption problem

- Consumer has CRRA utility: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.
- Consumer has two assets.
- Risk free: return r
- Equity: return $r + \pi$ and proportional variance σ^2.
- Consumer invests asset share θ in equities and consumes at rate c

$$dx = [(r + \theta\pi)x - c]dt + \theta\sigma x dz$$

so, $a(x, t) = [(r + \theta\pi)x - c]$ and $b(x, t) = \theta\sigma x$.

- Bellman Equation and Ito's Lemma:

$$\begin{aligned}\rho V(x,t)dt &= \max_{c,\theta}\{u(c)dt + E(dV)\} \\ & \max_{c,\theta}\left\{u(c)dt + \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt\right\}\end{aligned}$$

- For this problem, V doesn't depend *directly* on t (why?):

$$\rho V(x)dt = \max_{c,\theta}\left\{u(c)dt + \left[\frac{\partial V}{\partial x}[(r+\theta\pi)x - c] + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\theta\sigma x)^2\right]dt\right\}$$

- Boundary condition: $V(x) = \psi\frac{x^{1-\gamma}}{1-\gamma}$. So,

$$\rho\psi\frac{x^{1-\gamma}}{1-\gamma}dt = \max_{c,\theta}\left\{u(c)dt + \left[\psi x^{-\gamma}[(r+\theta\pi)x - c] - \frac{\gamma}{2}\psi x^{-\gamma-1}(\theta\sigma x)^2\right]dt\right\}$$

- First-order conditions:

$$\psi x^{-\gamma}\pi x - 2\left(\frac{\gamma}{2}\right)\psi x^{-\gamma-1}\theta(\sigma x)^2 = 0$$

$$u'(c) = c^{-\gamma} = \psi x^{-\gamma}$$

- Simplification implies:

$$\theta = \frac{\pi}{\gamma\sigma^2}$$

$$c = \psi^{-\frac{1}{\gamma}} x$$

- Plugging this back into last equation on previous page implies:

$$\psi^{-\frac{1}{\gamma}} = \frac{\rho}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\pi^2}{2\gamma\sigma^2}\right)$$

- Interesting case: $\gamma = 1$.

- $c = \rho x$. So $MPC \simeq 0.05$.

- $\theta = \frac{\pi}{\gamma\sigma^2} = \frac{0.06}{1 \times (0.16)^2} = 2.34$!

- How many households place 2.34 times their wealth (including human capital) in the stock market?

3 Application: Stopping problem

- Every instant the firm decides whether to continue and get instantaneous flow payoff, $w(x,t)dt$, or to stop and get termination payoff, $\Omega(x,t)$.

- We'll assume that $w(x,t)$ is increasing in x.

- Continuous time value function is given by (limit as $\Delta t \longrightarrow 0$):

$$V(x,t) = \max\left\{w(x,t)\Delta t + (1+\rho\Delta t)^{-1} EV(x',t'), \Omega(x,t)\right\}.$$

- Assume that

$$dx = a(x,t)dt + b(x,t)dz$$

- The solution to this problem is a stopping rule

$$\begin{array}{ll} \text{if} \quad x > x^*(t) & \text{continue} \\ \text{if} \quad x \leq x^*(t) & \text{stop} \end{array}$$

- Motivation: Irreversibly closing a production facility.

- Distinguish continuation region and stopping region of state space.

- The stopping region is the set of points $\langle x, t \rangle$ such that $x(t) \leq x^*(t)$.

- In continuation region, use Ito's Lemma to characterize the value function:

$$
\begin{aligned}
\rho V(x,t)dt &= w(x,t)dt + E(dV) \\
&= w(x,t)dt + \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right] dt
\end{aligned}
$$

- We need to solve this partial differential equation.

4 Value Matching

Recall that $\Omega(x,t)$ is the termination payoff.

One set of boundary conditions is

$$\lim_{x \to x^*(t)} V(x,t) = \Omega(x^*(t),t)$$

for all boundary points, $\langle x^*(t), t\rangle$.

This is referred to as a value matching condition (continuity of the value function at the boundary).

Heuristic proof. There exists a neighborhood $(x^*, x^* + \varepsilon)$ in which:

$$\begin{aligned} V(x,t) &= w(x,t)\Delta t + (1+\rho\Delta t)^{-1} EV(x', t') \\ &= w(x,t)\Delta t + (1+\rho\Delta t)^{-1} \left[\frac{V(x+b\sqrt{\Delta t}, t') + \Omega(x - b\sqrt{\Delta t}, t')}{2} \right] \\ &= \frac{1}{2} \lim_{x \downarrow x^*(t)} V(x,t) + \frac{1}{2}\Omega(x^*(t), t) + h.o.t. \\ &= \lim_{x \downarrow x^*(t)} V(x,t) + \frac{1}{2}\left[\Omega(x^*(t), t) - \lim_{x \downarrow x^*(t)} V(x,t) \right] + h.o.t. \end{aligned}$$

This implies that

$$\lim_{x \downarrow x^*(t)} V(x,t) = \lim_{x \downarrow x^*(t)} V(x,t) + \frac{1}{2}\left[\Omega(x^*(t), t) - \lim_{x \downarrow x^*(t)} V(x,t) \right],$$

which implies

$$\lim_{x \downarrow x^*(t)} V(x,t) = \Omega(x^*(t), t).$$

5 Smooth Pasting

We could now apply backwards induction techniques if we knew the "free boundary" $x^*(t)$.

To pin down the free boundary, we need another boundary condition, which is derived from optimization and called smooth pasting:

$$V_x(x^*(t), t) = \Omega_x(x^*(t), t).$$

Heuristic proof. Suppose slopes to left and right of stopping point are given by m and $m+\theta$, where $\theta \geq 0$. This is a convex kink. If you stop now you get payoff Ω. If you wait another instant and then stop, you get payoff:

$$w\Delta t + (1+\rho\Delta t)^{-1}\left[\Omega - \frac{mb\sqrt{\Delta t}}{2} + \frac{(m+\theta)b\sqrt{\Delta t}}{2}\right].$$

So the net payoff of waiting is

$$w\Delta t + (1+\rho\Delta t)^{-1}\left[\Omega - \frac{mb\sqrt{\Delta t}}{2} + \frac{(m+\theta)b\sqrt{\Delta t}}{2}\right] - \Omega$$
$$= w\Delta t + (1+\rho\Delta t)^{-1}\left[\Omega + \frac{\theta b\sqrt{\Delta t}}{2}\right] - \Omega$$

Multiply through by $(1+\rho\Delta t)$, simplify and remove terms of at least order Δt to find:

$$(1+\rho\Delta t)w\Delta t + \left[\Omega + \frac{\theta b\sqrt{\Delta t}}{2}\right] - (1+\rho\Delta t)\Omega = \frac{\theta b\sqrt{\Delta t}}{2}.$$

Intuition: At the boundary, the agent should be indifferent between continuation and stopping. If value functions don't smooth paste at $x^*(t)$, then stopping at $x^*(t)$ can't be optimal. Better to stop an instant later. If there is a (convex) kink at the boundary, then the gain from waiting is in $\sqrt{\Delta t}$ and the cost from waiting is in Δt. So there *can't* be a kink at the boundary. Hence:

$$V_x(x^*(t), t) = \Omega_x(x^*(t), t).$$

How would you rule out *concave kinks*?

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9.2 Application: Merton's Problem

9.3 Application: Stopping problem

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9.5 Boundary condition: Smooth Pasting

10.1 Stopping problem revisited

10.2 Solving second order ODE's

10.3 Stopping problem resolved

6 Stopping Problem Revisited

Assume that a continuous time stochastic process $x(t)$ is an Ito process,

$$dx = adt + bdz.$$

You might imagine that x is the price of a commodity produced by this firm. While in operation, the firm has flow profit

$$w(x) = x.$$

Assume that the firm can always costlessly (permanently) exit the industry and realize termination payoff

$$\Omega = 0.$$

Intuitively, this firm will have a stationary stopping rule,

$$\begin{array}{lll} \text{if} & x > x^* & \text{continue} \\ \text{if} & x \leq x^* & \text{stop (exit)} \end{array} .$$

Let $x' = x(t + \Delta t)$, and let ρ represent the interest rate. So

$$V(x) = \max\left\{x\Delta t + (1 + \rho\Delta t)^{-1} EV(x'), 0\right\}$$

In the stopping region,

$$V(x) = 0.$$

In the continuation region,

$$V(x) = x\Delta t + (1 + \rho\Delta t)^{-1} EV(x')$$

$$(1 + \rho\Delta t)V(x) = (1 + \rho\Delta t)x\Delta t + EV(x')$$

$$\rho V(x)\Delta t = (1 + \rho\Delta t)x\Delta t + EV(x') - V(x)$$

Multiply out and let $\Delta t \rightarrow 0$. Terms of order $dt^2 = 0$.

$$\rho V(x)dt = xdt + E(dV) \quad (*)$$

Now substitute in for $E(dV)$, using Ito's Lemma:

$$\begin{aligned} E(dV) &= \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} b(x,t)^2\right] dt \\ &= \left[aV' + \frac{b^2}{2} V''\right] dt. \end{aligned}$$

Substituting this expression into equation (*), we find

$$\rho V(x) dt = x dt + \left[aV' + \frac{b^2}{2} V''\right] dt.$$

which is a second-order ordinary differential equation,

$$\rho V = x + aV' + \frac{b^2}{2} V''.$$

What are our boundary conditions?

Value matching:

$$V(x^*) = 0.$$

Smooth pasting:

$$V'(x^*) = 0.$$

As $x \to \infty$ the option value of exiting goes to zero, so V converges to the value associated with the policy of never exiting the industry. Hence,

$$\lim_{x \to \infty} \frac{V(x)}{\frac{1}{\rho}\left(x + \frac{a}{\rho}\right)} = 1.$$

We'll derive this equation later. We could also have written

$$\lim_{x \to \infty} V'(x) = \frac{1}{\rho}$$

7 Solving second order ODE's

To solve the differential equations that come up in economics, it is helpful to recall a few general results from the theory of differential equations.

Consider a generic second order ordinary differential equation:

$$F''(x) + A(x)F'(x) + B(x)F(x) = C(x)$$

This equation is referred to as the "complete equation." Note that $A(x)$, $B(x)$, and $C(x)$, are given functions. We are trying to solve for F with independendent variable x.

Now consider a "reduced equation" in which $C(x)$ is replaced by 0.

$$F''(x) + A(x)F'(x) + B(x)F(x) = 0$$

Solving this reduced differential equation will enable us to solve the complete equation. We begin by characterizing the solution of the reduced equation.

Theorem 7.1 *Any solution, $\hat{F}(x)$, of the reduced equation can be expressed as a linear combination of any two solutions of the reduced equation, F_1 and F_2, that are linearly independent.*

$$\hat{F}(x) = C_1 F_1(x) + C_2 F_2(x)$$

Note that two solutions are linearly independent if there do *not* exist constants A_1 and A_2 such that

$$A_1 F_1(x) + A_2 F_2(x) = 0 \qquad \text{for all } x$$

Theorem 7.2 *The general solution of the complete equation is the sum of any particular solution of the complete equation and the general solution of the reduced equation.*

8 Stopping problem resolved

Recall the differential equation that characterizes the continuation region

$$\rho V = x + aV' + \frac{b^2}{2}V'' \qquad \text{(complete equation)}$$

Consider the reduced equation,

$$0 = -\rho V + aV' + \frac{b^2}{2}V'' \qquad \text{(reduced equation)}$$

Our first challenge is to find solutions of this equation. Consider the class,

$$e^{rx}$$

To confirm that this is in fact a solution, differentiate and plug in to find

$$0 = -\rho e^{rx} + are^{rx} + \frac{b^2}{2}r^2 e^{rx}.$$

This implies that

$$0 = -\rho + ar + \frac{b^2}{2} r^2.$$

Apply quadratic formula,

$$r = \frac{-a \pm \sqrt{a^2 + 2b^2\rho}}{b^2}$$

Let r^+ represent the positive root and let r^- represent the negative root. Any solution to the reduced equation can be expressed

$$C^+ e^{r^+ x} + C^- e^{r^- x} \qquad \text{(general solution to the reduced equation)}$$

The general solution to the complete equation can be expressed as the sum of a particular solution to the complete differential equation and the general solution to the reduced equation.

To find a particular solution, consider the payoff function of the policy "never leave the industry." The value of this policy is

$$E \int_0^\infty e^{-\rho t} x(t) dt.$$

Note that

$$\begin{aligned} E\left[x(t)\right] &= E\left[x(0) + \int_0^t dx(t)\right] \\ &= E\left[x(0) + \int_0^t \left[a dt + b dz(t)\right]\right] \\ &= E\left[x(0) + at + \int_0^t b dz(t)\right] \\ &= x(0) + at \end{aligned}$$

The value of the policy "never leave" is derived with integration by parts:

$$\begin{aligned} E\int_0^\infty e^{-\rho t}x(t)dt &= \int_0^\infty e^{-\rho t}\left[x(0)+at\right]dt \\ &= \frac{x(0)}{\rho}+\left[-\frac{1}{\rho}e^{-\rho t}at\right]_0^\infty - \int_0^\infty -\frac{1}{\rho}e^{-\rho t}a dt \\ &= \frac{1}{\rho}\left(x(0)+\frac{a}{\rho}\right) \end{aligned}$$

Hence, our (candidate) particular solution takes the form

$$V(x)=\frac{1}{\rho}\left(x+\frac{a}{\rho}\right).$$

Confirm that this is a solution to the complete differential equation.

$$\rho\cdot\left[\frac{1}{\rho}\left(x+\frac{a}{\rho}\right)\right]=x+a\cdot\frac{1}{\rho}+\frac{b^2}{2}\cdot 0 \qquad \checkmark$$

We now can draw all of these pieces together. First, we have our general solution to the reduced equation

$$C^{+}e^{r^{+}x} + C^{-}e^{r^{-}x}.$$

where the roots are given by

$$r^{+}, r^{-} = \frac{-a \pm \sqrt{a^2 + 2b^2\rho}}{b^2}$$

Second, we have our particular solution:

$$\frac{1}{\rho}\left(x + \frac{a}{\rho}\right).$$

So the general solution of the complete equation is

$$V(x) = \frac{1}{\rho}\left(x + \frac{a}{\rho}\right) + C^{+}e^{r^{+}x} + C^{-}e^{r^{-}x}$$

Finally, we have boundary conditions.

Value matching:

$$V(x^*) = 0.$$

Smooth pasting:

$$V'(x^*) = 0.$$

We know

$$\lim_{x\to\infty} V'(x) = \frac{1}{\rho},$$

which implies $C^+ = 0$. Value matching and smooth pasting imply

$$V(x^*) = \frac{1}{\rho}\left(x^* + \frac{a}{\rho}\right) + C^- e^{r^- x^*} = 0 \tag{1}$$

$$V'(x^*) = \frac{1}{\rho} + r^- C^- e^{r^- x^*} = 0. \tag{2}$$

Equation (1) implies

$$C^- e^{r^- x^*} = -\frac{1}{\rho}\left(x^* + \frac{a}{\rho}\right).$$

Plugging this expression into equation (2) implies

$$\frac{1}{\rho} - \frac{r^-}{\rho}\left(x^* + \frac{a}{\rho}\right) = 0$$

Hence,

$$x^* = \frac{1}{r^-} - \frac{a}{\rho}.$$

We have,

$$r^- = \frac{-a - \sqrt{a^2 + 2b^2\rho}}{b^2}.$$

Hence,

$$x^* = -\frac{b^2}{a + \sqrt{a^2 + 2b^2\rho}} - \frac{a}{\rho} < 0$$

Some interesting special cases (see problem set):

$$
\begin{aligned}
x^*_{(a=0)} &= -\frac{b}{\sqrt{2\rho}} < 0 \\
\lim_{a \longrightarrow \infty} x^* &= -\infty \\
\lim_{a \longrightarrow -\infty} x^* &= 0 \\
\lim_{b \longrightarrow 0} x^* &= \left\{ \begin{array}{ll} -\frac{a}{\rho} & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{array} \right\} \\
\lim_{b \longrightarrow \infty} x^* &= -\infty \\
\lim_{\rho \longrightarrow 0} x^* &= \left\{ \begin{array}{ll} -\infty & \text{if } a \geq 0 \\ \frac{b^2}{2a} & \text{if } a < 0 \end{array} \right\} \\
\lim_{\rho \longrightarrow \infty} x^* &= 0
\end{aligned}
$$

Illustrative calibration:

$$
\begin{aligned}
x^*_{(a=0)} &= -\frac{b}{\sqrt{2\rho}} \\
&= -\frac{b}{\sqrt{2 \times .02}} \\
&= -5b
\end{aligned}
$$

Wait until the x gets 5 standard deviations below the break-even threshold ($x = 0$) before shutting down.

Finally, now that we have our solution, it is easy to calculate the option value of stopping:

$$O(x) = V(x) - \frac{1}{\rho}\left(x + \frac{a}{\rho}\right)$$

Substituting in for $V(x)$ yields

$$O(x) = \left\{ \begin{array}{ll} C^{-}e^{r^{-}x} & \text{if } x \geq x^{*} \\ -\frac{1}{\rho}\left(x + \frac{a}{\rho}\right) & \text{if } x < x^{*} \end{array} \right\}$$

So as $x \to \infty$ the option value of stopping goes to zero.

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