Bridging the gap between lattice models and TQFTs

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Lattice models and TQFTs

2D gapped lattice model
Lattice models and TQFTs

2D gapped lattice model \rightarrow \text{TQFT} \hspace{1cm} \text{anyon theory}
Data for (Abelian) anyon theories

- Anyon types: $\mathcal{C} = \{a, b, c, \ldots\}$

- Fusion rules: $a \times b = ab$

- Fusion/braiding data: $F(a, b, c), \ R(a, b) \in U(1)$
Main question: What is microscopic definition of anyon data?
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Focus of this talk: $F(a, b, c)$  ‘$F$-symbol’
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Motivation:

• Poorly understood quantity
• Application to anomaly computation
Warm up: microscopic definition of $R(a,a)$

$$R(a, a) = \text{exchange statistics of } a$$

Naive definition:

$$|\Psi\rangle \rightarrow R(a, a)|\Psi\rangle$$
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Naive definition:

$$|\Psi\rangle \rightarrow R(a, a) |\Psi\rangle$$

Non-universal phases do not cancel
Better definition:

\[ M_3 M_2 M_1 |\Psi\rangle = R(a, a) \cdot M_1 M_2 M_3 |\Psi\rangle \]
F-symbol: abstract picture

\[ |1\rangle \quad \text{and} \quad |2\rangle \]
F-symbol: abstract picture

\[ |1\rangle = F(a, b, c) \cdot |2\rangle \]
\[ F(a, b, c) \text{ well-defined up to:} \]

\[ F(a, b, c) \rightarrow F(a, b, c) \cdot \frac{e^{i\nu(a,b)}e^{i\nu(ab,c)}}{e^{i\nu(b,c)}e^{i\nu(a,bc)}} \]

‘gauge transformations’
Microscopic definition of F-symbol

Arrange anyons along line:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
| & | & |
\end{array}
\]

\[
\begin{array}{c}
a & b & c \\
x & x' & x''
\end{array}
\]

Label as: \(|a_x, b_{x'}, c_{x''}, \ldots\rangle\)
Movement operator: $M_{x',x}^{a}$
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$M_{x',x}^a |\ldots a_x \ldots\rangle \propto |\ldots a_{x'} \ldots\rangle$
Movement operator: $M^a_{x', x}$

$M^a_{x', x} |...a_x...\rangle \propto |...a_{x'}...\rangle$

Splitting operator: $S(a, b)$
Movement operator: $M_{x',x}^a$

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Splitting operator: $S(a, b)$
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Splitting operator: $S(a, b)$

$$S(a, b)|... (ab)_1 ...\rangle \propto |...a_1, b_2...\rangle$$
\[ |1\rangle = F(a, b, c) \cdot |2\rangle \]
\[ |1\rangle = M_{12}^b \cdot M_{01}^a \cdot S(a, b) \cdot M_{32}^c \cdot S(ab, c) \left| (abc)_{1} \right\rangle \]

\[ |2\rangle = M_{32}^c \cdot S(b, c) \cdot M_{12}^{bc} \cdot M_{01}^a \cdot S(a, bc) \left| (abc)_{1} \right\rangle \]
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**Phase ambiguity:**

\[ S(a, b) \rightarrow e^{i\phi(a,b)} \cdot S(a, b), \quad M_{x'x}^a \rightarrow e^{i\theta_{x'x}(a)} \cdot M_{x'x}^a \]
\[ |1\rangle = M_{12}^b \cdot M_{01}^a \cdot S(a,b) \cdot M_{32}^c \cdot S(ab,c) \ |(abc)_1\rangle \]
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Effect on \( F \):

\[ F(a,b,c) \rightarrow F(a,b,c) \cdot \frac{e^{i\nu(a,b)} e^{i\nu(ab,c)}}{e^{i\nu(b,c)} e^{i\nu(a,bc)}} \]

\[ \nu(a,b) = \phi(a,b) + \theta_{12}(b) \]
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Example: Toric code model

\[ H = - \sum_{\text{+}} \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x - \sum_{\text{square}} \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z \]

(Kitaev, 2003)
Example: Toric code model

\[ H = - \sum_{\pm} \sigma^x_1 \sigma^x_2 \sigma^x_3 \sigma^x_4 - \sum_\square \sigma^z_1 \sigma^z_2 \sigma^z_3 \sigma^z_4 \]
Example: Toric code model

\[ M^e = \sigma^z \]
\[ S(e, e) = \sigma^z \quad \Longrightarrow \quad F(e, e, e) = 1 \]

(Kitaev, 2003)
Remarks

- Can also give microscopic definition of $R(a, b)$

  $\Rightarrow$ can extract complete set of anyon data from lattice model

- Application: computing anomalies from SPT edge theories

  (see Kyle Kawagoe’s poster!)