Categorical symmetry – a holographic view of symmetry

Xiao-Gang Wen (MIT), June, 2020

Kong-Wen arXiv:1405.5858
Ji-Wen arXiv:1912.13492

Simons Collaboration on Ultra-Quantum Matter
Symmetry in quantum systems

- What is a **symmetry**?
  - A **bosonic symmetry** is a set of linear constraints on local Hamiltonians of **bosonic systems** (ie qubit systems).
  - A **fermionic symmetry** is a set of linear constraints on local Hamiltonians of **fermionic systems**.
- **Linear constraints**: $W_a H = HW_a$, where $a$ labels different symmetry transformations. It can be (1) different group elements, and (2) different loops, close surfaces, etc. where the operator act, ... ...
- If the transformation acts on the whole space → the usual global symmetry (0-symmetry).
- If the trans. acts on all the codimension-$k$ closed sub-spaces → the $k$-symmetry.

- **Symmetry** = a class of Hamiltonians $\{ H \mid W_a H = HW_a \}$, which are called the **symmetric Hamiltonians**
View symmetry via their **charged excitations**

- **A quantum system** (a 2d spin-1/2 system) is determined by Hamiltonian $H = - \sum_i Z_i$ and deformation class $\{\delta H\}$.

- **An excitation** = something can be trapped: $H$ has a gap $H + \delta H_{\text{trap}}$ also has a gap $\rightarrow$ trapped states: $|\downarrow\rangle$, $|\downarrow\downarrow\rangle$, ... ...

- **Type**: (*determined by $H$ and deformation class $\{\delta H\}$*)
  - Without symmetry ($\{\delta H\}$ are formed by local operators)
    $|\downarrow\rangle$, $|\downarrow\downarrow\rangle$ can be deformed into $|0\rangle$ without closing the gap.
    $|\downarrow\rangle \sim |\downarrow\downarrow\rangle \sim |0\rangle$ the same trivial type $1$.
  - With $Z_2$-symm. $U = \prod_i Z_i$ ($\{\delta H \mid \delta H U = U \delta H\}$)
    $|\downarrow\downarrow\rangle$ and $|0\rangle$ are of the same trivial type $1$ ($Z_2$-charge-0).
    $|\downarrow\rangle$ has a different type $e$ ($Z_2$-charge-1).
The charge excitations form a fusion 2-category

- With the $Z_2$-symmetry, the excitations have a fusion rule
  \[ 1 \otimes 1 = 1, \quad 1 \otimes e = e \otimes 1 = e, \quad 1 \text{ is the fusion unit} \]
  \[ e \otimes e = 1, \quad e \text{ has mod 2 conservation} \]
- \( \{1, e\} \) generate a fusion 2-category \( 2\text{Rep}(Z_2) \).

- In 2-dimensional space, we have
  string-like excitations \( \rightarrow \) objects (\( \{1_s, e_s\} \) trivial, descendent),
  point-like excitations \( \rightarrow \) 1-morphisms (\( \{1, e\} \) generators).

2 layers \( \rightarrow \) a fusion 2-category.

- \( \{1, e\} \) are the generators of the fusion 2-category.

**Tannaka duality**: symmetric fusion category \( \leftrightarrow \) symm. group

**Categorical view of symmetry**:

The class of 2d $Z_2$-symmetric Hamiltonians:
\[ \{H_{Z_2}\} \]
\[ = \text{The class of fusion-preserveing Hamiltonians describing interacting excitations in } 2\text{Rep}(Z_2): \{H_{2\text{Rep}(Z_2)}\}. \]

(The class of Hamiltonians describing particles with mod 2 conservation. No longer think about symm. transformations)
Use entanglement to simulate symmetry

- **$Z_2$-symmetry in 2d = Fusion 2-category $\mathcal{2Rep}(Z_2)$**
  - $\mathcal{2Rep}(Z_2)$ describes topo. excitations from a topo. order without symm. *But which topo. order has excitations $\{1, e\}$?*

- **The fusion 2-category $\{1, e\} = \mathcal{2Rep}(Z_2)$ describes the excitations on a boundary of 3d $Z_2$ gauge theory**
  - The excitations in 3d $Z_2$-gauge theory is generated by point-like excitations $e$ (the bosonic $Z_2$ charge) and string-like excitations $s$ (the bosonic $Z_2$-flux string)
  - Bulk fusion rule: $e \otimes e = 1, \quad s \otimes s = 1_s$ (trivial string)

- **The boundary is induced by the $Z_2$-flux loops condensation. The boundary excitations are described by $\{1, e\} = \mathcal{2Rep}(Z_2)$.**

**Entanglement (topological order) in one higher dimension → symmetry**

- **The class of 2d $Z_2$-symm. Hamiltonians: $\{H_{Z_2}\} =$ Boundary Hamiltonians of 3d $Z_2$ gauge theory ($\infty$ bulk gap): $\{H_{Z_2}^{\text{bndry}}\}$**
Boundary symmetry from bulk symmetry

- The mod 2 conservation of the bulk-\( e \) →
  The \( Z_2 \) 0-symmetry in the bulk →
  The mod 2 conservation of boundary-\( e \) →
  The \( Z_2 \) symmetry at the boundary.

\[
\{1, e\} = 2\text{Rep}(Z_2)
\]

String condensed boundary = trivial \( Z_2 \)-symmetric phase
\( Z_2 \) 0-symmetry is not broken, \( Z_2^{(1)} \) 1-symmetry is broken.

- But the bulk actually has a bigger symmetry – \( Z_2 \times Z_2^{(1)} \)
Categorical symmetry. The mod 2 conservation of the bulk-\( s \) → The \( Z_2^{(1)} \) 1-symmetry in the bulk.

Charge condensed boundary = \( Z_2 \)-symm. breaking phase
\( Z_2 \) 0-symmetry is broken, \( Z_2^{(1)} \) 1-symmetry is not broken.

- The \( Z_2 \)-symmetry breaking phase has excitations
  \[
  \{1, s\} = 2\text{Vec}_{Z_2}.
  \]
  \( s \) is the string-like symmetry breaking domain walls.) The 3d \( Z_2 \) gauge theory has a \( Z_2 \)-charge condensed boundary with excitations
  \[
  \{1, s\} = 2\text{Vec}_{Z_2}.
  \]
The categorical symmetry is not quite $\mathbb{Z}_2 \times \mathbb{Z}_2^{(1)}$. The particle $e$ and the string $s$ in bulk have a non-trivial mutual statistics. So we denote the categorical symmetry as $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$.

A 2d $\mathbb{Z}_2$-symmetric system actually have a bigger symmetry – the $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$ Categorical symmetry.

The class of 2d $\mathbb{Z}_2$-symmetric Hamiltonians: $\{H_{\mathbb{Z}_2}\}$

= The Hamiltonians for excitations $2\mathcal{R}ep(\mathbb{Z}_2)$: $\{H_{2\mathcal{R}ep(\mathbb{Z}_2)}\}$

= Boundary of 3d $\mathbb{Z}_2$ gauge theory ($\infty$ bulk gap): $\{H_{\text{bndry } \mathbb{Z}_2\text{-gauge}}\}$

= The class of Hamiltonians with $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$ symm. $\{H_{\text{bndry } \mathbb{Z}_2\text{-gauge}}\}$

All phases of $\mathbb{Z}_2$-symmetric systems

= All phases of $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$-symmetric systems

All boundary phase of 3+1D $\mathbb{Z}_2$ gauge theory w/ $\infty$ bulk gap.
$\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$ categorical symmetry at critical point

- **A holographic view** on symmetry breaking transition:
  
  A $\mathbb{Z}_2$ symmetric system actually has a bigger $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$ categorical symmetry

  $\mathbb{Z}_2$ SSB $\Rightarrow$ $\mathbb{Z}_2$ Symmetric

  $\mathbb{Z}_2^{(1)}$ Symmetric $\Leftrightarrow$ $\mathbb{Z}_2^{(1)}$ SSB

  **coupling**

- **The critical point at the symmetry-breaking transition** has the full (unbroken) $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$ categorical symmetry.

  The critical point $\equiv$ The minimal non-condensing boundary of the 3+1D $\mathbb{Z}_2$ gauge theory
States with full categorical symm. must be gapless

For a 2d system with $Z_2 \lor Z_2^{(1)}$ categorical symmetry:

- its gapped phases must either break the $Z_2$ 0-symmetry or the $Z_2^{(1)}$ 1-symmetry, but not both.

- The phase with the full $Z_2 \lor Z_2^{(1)}$ categorical symmetry must be gapless.

(The 1d version of the above results were proved by Levin arXiv:1903.09028)
Apply categorical symmetry to AdS/CFT duality


- **Witten**: “for gauge theory, suppose the AdS theory has a gauge group $G$, [...] Then in the scenario of [13], the group $G$ is a global symmetry group of the conformal field theory on the boundary.”

- **G**-symm.-breaking-transition CFT has a categorical symm. described by the $G$-gauge theroy in one higher dimension, *which uniquely determines the bulk theory* $\rightarrow$ A proposal: Pure $G$-gauge theory (w/ charge fluc. & gravity) in $(n + 1)$-dim. AdS space $=$ CFT at the $G$-symm.-breaking-transition in $n$-dim. space, not other CFT’s with $G$-symmetry.

Xiao-Gang Wen (MIT), June, 2020

Ji-Wen arXiv:1912.13492
Apply categorical symm. to 3+1D $\mathbb{Z}_2$ gauge theory

- Phase transitions from $\mathbb{Z}_2$ topological order (described by $\mathbb{Z}_2$ gauge theory) to trivial product state:
  - Higgs transition induced by $\mathbb{Z}_2$ point-charge condensation. 
    \textit{(same as 3+1D Ising transition)}
    The Ising CFT has a $\mathbb{Z}_2$ symmetry.
  - Confinement transition induced by $\mathbb{Z}_2$ flux-string condensation.
    The transition is first order $\rightarrow$ a critical point.
    The confinement CFT has a $\mathbb{Z}^{(1)}_2$ symmetry.

- The Ising CFT $= \text{The confinement CFT} \ ???$ 
  They might have the same emergent symmetries.

- The Ising CFT has categorical symmetry $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(2)}$
  The confinement CFT has categorical symmetry $\mathbb{Z}_2^{(1)} \vee \mathbb{Z}_2^{(1)}$

Fradkin-Shenker, PRD \textbf{19} 3682 (79)
Symmetry $G$ selects a class of symmetric Hamiltonians $\{H_G\}$.

- Gravitational anomaly (denoted by $M$) also selects a class of (effective) Hamiltonians $\{H_M\}$ that have the same anomaly $M$.
- $\{H_M\}$ is a linear space: $H_M + H'_M \in \{H_M\}$.
- Gravitational anomaly is also a set on linear constraints, just like symmetry.
Symmetry $G$ selects a class of symmetric Hamiltonians $\{H_G\}$.

- Gravitational anomaly (denoted by $M$) also selects a class of (effective) Hamiltonians $\{H_M\}$ that have the same anomaly $M$.
- $\{H_M\}$ is a linear space: $H_M + H'_M \in \{H_M\}$.
- Gravitational anomaly is also a set on linear constraints, just like symmetry.

Every 0-symmetry, higher symmetry, or even the most general algebraic higher symmetry corresponds to a gravitational anomaly. (Many-to-one correspondence)

Due to this connection, we also refer gravitational anomaly as categorical symmetry.

- Some categorical symmetries (gravitational anomalies) may not correspond to a symm., not even the algebraic higher symm.
What is gravitational anomaly?

- **Traditionally**, an gravitational anomaly is a non-invariance of the path integral measure under a diffeomorphism transformation of the spacetime.

- **Recent point of view**: A fusion $n$-category $\mathcal{C}$ describing the excitations in $n$-dimensional space (i.e., a field theory) has a gravitational anomaly if $\mathcal{C}$ cannot be realized by excitations in a gapped phase of a bosonic lattice model in the same dimension and without symmetry.

- **Categorical view of symmetry**: We use fusion 2-category $\mathcal{2}\text{Rep}(\mathbb{Z}_2) = \{1, e\}$ to describe 2d $\mathbb{Z}_2$ symmetry.
  
  *ie we ignore the symmetry, and use fusion 2-category $\mathcal{2}\text{Rep}(\mathbb{Z}_2) = \{1, e\}$ and their arbitrary interactions (preserve fusion) to obtain the class of Hamiltonians $\{H_{\mathcal{2}\text{Rep}(\mathbb{Z}_2)}\}$ → the class of Hamiltonians selected by the $\mathbb{Z}_2$ symm. $\{H_{\mathbb{Z}_2}\}$."

- $\mathcal{2}\text{Rep}(\mathbb{Z}_2) = \{1, e\}$ has gravitational anomaly, since it cannot be realized by 2d bosonic lattice model without symmetry.
What REALY is gravitational anomaly?

- **A conjecture**: A fusion $n$-category $\mathcal{C}$ can always be realized by the excitations on a certain boundary of a bosonic lattice model in one higher dimension (ie in $(n + 1)$-dim. space).
  - An anomaly-free fusion $n$-category $\mathcal{C}$ can be realized by the excitations on a certain boundary of a product state.
  - An anomalous fusion $n$-category $\mathcal{C}$ can be realized by the excitations on a certain boundary of a non-trivial anomaly-free topological order $\mathcal{M}$ in one higher dimension.

- **Holographic principle** A boundary fusion $n$-category $\mathcal{C}$ uniquely determines a bulk topological order $\mathcal{M} = \text{bulk}(\mathcal{C})$. Kong-Wen arXiv:1405.5858; Kong-Wen-Zheng arXiv:1702.00673

  **Gravitational anomaly of $\mathcal{C} = \text{bulk topological order} \mathcal{M}**

  Those anomalies are often non-invertible. Ji-Wen arXiv:1905.13279

- **Symmetry** $(2\text{Rep}(\mathbb{Z}_2)) \rightarrow$ Gravitational anomaly = Bulk topo. order (bulk $\mathbb{Z}_2$ gauge theory) = categorical symm. $(\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)})$
Emergent categorical symmetry

- Consider a **field theory** in \( n \)-dimensional space with low energy excitation described by a fusion \( n \)-category \( C \).
  - We assume higher energy excitations always have very high energies, and ignore them.
  - Here **field theory** means a theory whose UV regularization is not specified. *When we say a field theory has a property, we mean that there exists a UV regularizations for the field theory, and such regularized field theory has the property.* (There are may be other different UV regularizations where regularized field theory does not the property).
  - Some excitations in \( C \) may come from symmetry charge (e.g. \( 2\text{Rep}(Z_2) \)), and other are topological. But the distinction is not important, and we pretend all excitations are topological excitations.
- **The emergent category symmetry in** \( C \) **is given by a topological order in one higher dimension** \( M = \text{bulk}(C) \).
Emergent categorical symmetry

→ Low energy properties

The emergent categorical symmetry $M$ controls the low energy properties of those excitations $C$ (such as their condensations)

- All the gapped and gapless phases formed by the low energy excitations $C$ have the same categorical symmetry $M$ (i.e., the same gravitational anomaly), which may be partially spontaneously broken.

- **Constraint on possible phases**: The excitations $C'$, in a gapped phase formed by condensing excitations in $C$, must satisfy $\text{bulk}(C') = \text{bulk}(C) \rightarrow$ The usual anomaly matching.

- The new insight here is that we can combine (higher, algebraic, higher, 't Hooft anomalous, etc) symmetries and (low energy) gravitational anomalies into an **effective gravitational anomaly** → the emergent categorical symmetry.
Emergent categorical symmetry → duality relation

- **Duality relation**: two gapped field theories with low energy excitations $C$ and $C'$ are **dual-equivalent** if and only if they have the same **categorical symmetry** $\text{bulk}(C) = \text{bulk}(C')$.

- **Dual-equivalent**: the states formed by $C$ have a one-to-one correspondence with the states formed by $C'$.

- In $n+1$D, pure $SU(N)$ gauge theory, $Z_N^{(1)}$ bosonic models, $Z_N^{(n-2)}$ bosonic models are **dual-equivalent**:

$$M = Z_N^{(1)} \vee Z_N^{(n-2)}.$$

- **Duality relation**: two symmetries $\mathcal{R}$ and $\mathcal{R}'$ (charge objects) are **dual-equivalent** if and only if they have the same **categorical symmetry** $\text{bulk}(\mathcal{R}) = \text{bulk}(\mathcal{R}')$.

- An 1d anomalous $Z_2^3$ symm and $D_4$ symm are **dual-equivalent**.
A theory for most general anomaly-free symmetry

- **Anomaly-free symmetry** allows **trivial symmetric phase** (an unique ground state on closed space of any topology)
- The *set* of **charge objects**, $\mathcal{R}$, of an anomaly-free symmetry is the *set* of **types** of excitations on a **trivial symmetric state**
- The charge objects of a symmetry in $n$-dimensional space form a fusion $n$-category, called the **representation category** $\mathcal{R}$.
  - For $n d$ bosonic 0-symmetry $G$, its charge objects (the particles carrying representations of $G$) generate the so called fusion $n$-category $\mathcal{R} = n\text{Rep}(G)$.
  - For $n d$ bosonic $(n-1)$-symmetry $Z_2^{(n-1)}$, its charge objects (the $(n-1)$-dimensional domain wall) generate the so called fusion $n$-category $\mathcal{R} = n\text{Vec}_{Z_2}$.
  - $\mathcal{R} = n\text{Vec}$ describes excitations on bosonic product state without bosonic symmetry
  - $\mathcal{R}_f = n s\text{Vec} \sim n\text{Rep}(Z_2)$ describes excitations on fermionic product state without fermionic symmetry

Xiao-Gang Wen (MIT), June, 2020
\( \mathcal{R} \) is a special kind of fusion \( n \)-category. \textit{Which kind?}

- A fusion \( n \)-category describes the charge objects of a symmetry if it can be reduced to the fusion \( n \)-category of no symmetry (via explicit symmetry breaking) \(\rightarrow\) Math definition:

\[
\text{Local fusion category} \quad (\mathbb{B}: \beta : \mathcal{R} \xrightarrow{\text{top}} n\text{Vec}. \mathbb{F}: \beta : \mathcal{R}_f \xrightarrow{\text{top}} n_s\text{Vec})
\]
defines \textbf{algebraic higher symmetry} (a most general symmetry?)

- A conjecture
  
  The class of Hamiltonians for int. excitations in \( \mathcal{R} \):
  \( \{H_{\mathcal{R}}\} \)
  \(=\) The class of symmetric Hamiltonians obtained from a set of linear constraints:
  \[
  \{H_{\mathcal{W}_a} \mid \mathcal{W}_a H = H\mathcal{W}_a\}
  \]

\textbf{Local fusion categories classify algebraic higher higher symmetries}
Algebraic higher symmetry $n\text{Vec}_G$

- Consider a $G$ symmetry breaking state in $nd$. The domain walls are labeled by $h \in G$ with fusion $h \otimes h' = hh' \rightarrow$ fusion $n$-category $n\text{Vec}_G$. Is $n\text{Vec}_G$ a local fusion category? Does $n\text{Vec}_G$ describe a symm?

- The fusion of the domain walls $\rightarrow$ a conservation law $\rightarrow$ a “symmetry”. But the symmetry breaking state is not a product state, and the domain walls may not be the charge objects. Also what is the symmetry transformations?

- A lattice model on a triangulation of a $n$-dimensional space. The degrees of freedom on the links $ij$ are given by $g_{ij} \in G$

$$H = -J \sum_{ij} \delta(g_{ij}) - B \sum_{i} \sum_{h \in G} Q_h(i) - U \sum_{ijk} \delta(g_{ij}g_{jk}g_{ik}^{-1}),$$

$$Q_h(i)|\cdots, g_{ij}, g_{ki}, \cdots\rangle = |\cdots, hg_{ij}, g_{ki}h^{-1}, \cdots\rangle, \quad g_{ij} = g_{ji}^{-1}.$$

- **Algebraic $(n-1)$-symmetry** is generated by Wilson loop operators $W_q(S^1) = \text{Tr} \prod_{ij \in S^1} R_q(g_{ij})$, $R_q$ is a rep. of $G$
Algebraic higher symmetry $n\mathcal{Vec}_G$

- The ground state of the bosonic Hamiltonian $H$ is a product state $|0\rangle = \otimes_{ij} |g_{ij} = 1\rangle$.
- The ground state is symmetric, since the action of the symmetry transformations on the ground subspace is proportional to identity $W_q(S^1)|0\rangle = \text{dim}(R_q)|0\rangle$.
- A $(n-1)$-dimensional excitation $h$ on top of the the ground state: changing $g_{ij} = 1$ to $g_{ij} = h$ on a $(d-1)$-dimensional closed subspace = a charge object of the Algebraic $(n-1)$-symmetry.
- Measure charge $W_q(S^1)|h\rangle = \text{Tr} R_q(h)|h\rangle >$. $h$ and $h' = ghg^{-1}$ carry the same charge.
- The symmetry satisfies $W_{q_1}(S^1)W_{q_2}(S^1) = \sum_{q_3} N_{q_1q_2}^{q_3} W_{q_3}(S^1)$, which is not a group algebra for non-Abelian $G$.

The $(n-1)$-dimensional excitations form a local fusion $n$-category $n\mathcal{Vec}_G$, which describes the algebraic $(n-1)$-symmetry generated by the Wilson loops $W_q(S^1)$.
A gapless state is very special, and has a lot of emergent symmetries. The full emergent symmetry may be the categorical symmetry

- A categorical symmetry (= grav. anomaly) is fully characterized by a topological order in one higher dimension.

- Categorical symm. (= grav. anomaly = bulk topo. order) may completely determines the minimal gapless state.

We can classify all gapped liquid phases in systems with a categorical symmetry. Such a classification includes

- SETs with algebraic higher symm.
- SPTs with algebraic higher symm.
- Gauge the algebraic higher symm.
- Anomalous algebraic higher symm.