The Price of Anarchy in Auctions

Part I: Introduction and Motivation

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Single-item Auction Problem:

Given:

- one item for sale.
- n bidders (with unknown private values for item, $v_1, \ldots, v_n$)
- Bidders' objective: maximize *utility* $=$ value $-$ price paid.

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Possible Auction Objectives:

- Maximize *social welfare*, i.e., the value of the winner.
- Maximize *seller revenue*, i.e., the payment of the winner.

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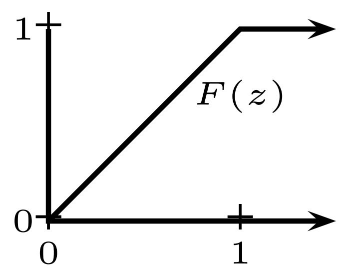

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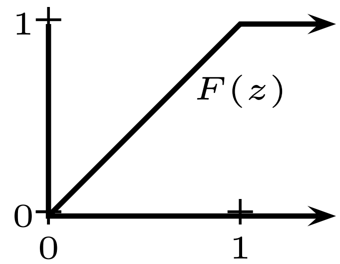

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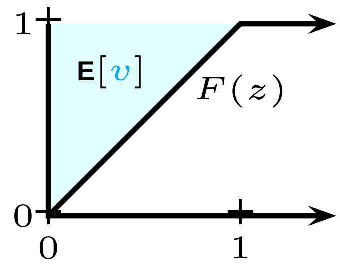

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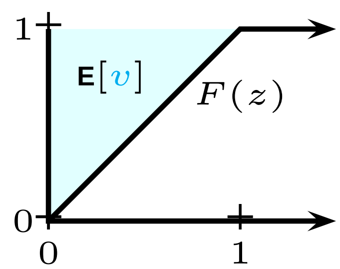

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Conclusion 2: bidder with highest value wins

Conclusion 3: first-price auction maximizes social welfare!

Questions?

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Definition: a *strategy profile* is in *Bayes-Nash Equilibrium (BNE)* if for all i, $b_i(v_i)$ is best response when others play $b_j(v_j)$ and $v_j \sim F_j$.

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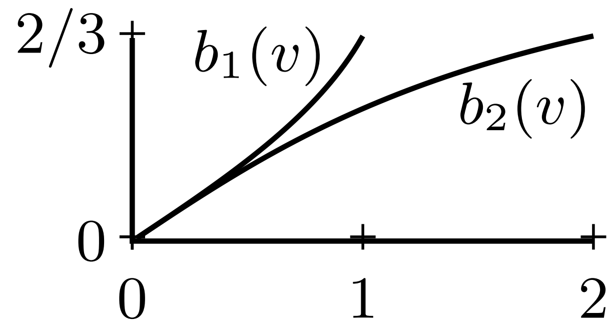

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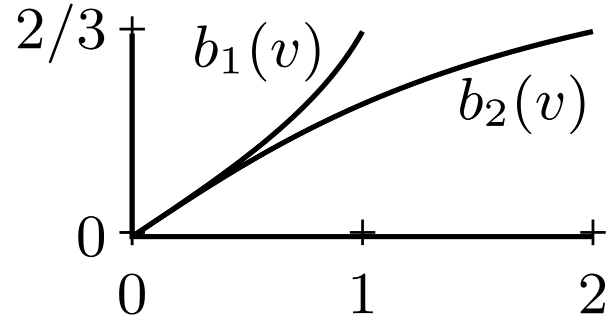

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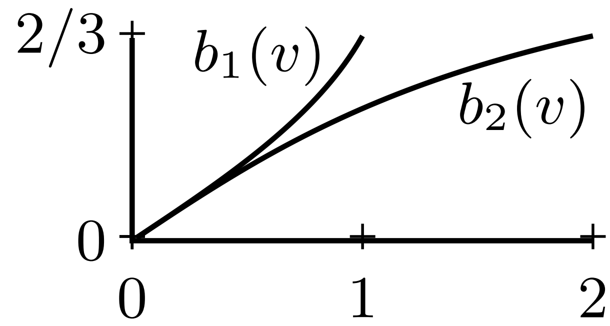

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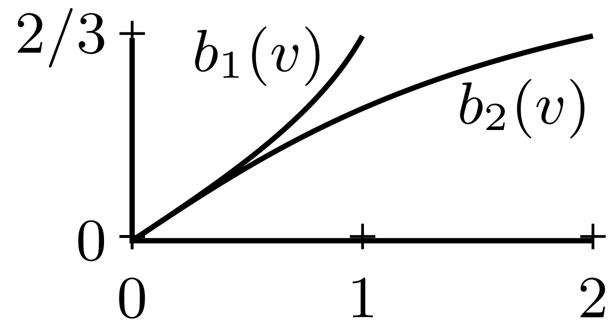

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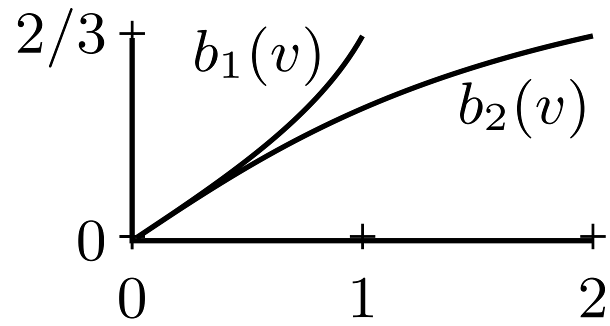

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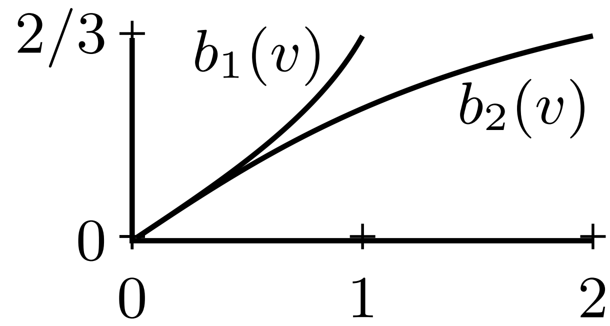

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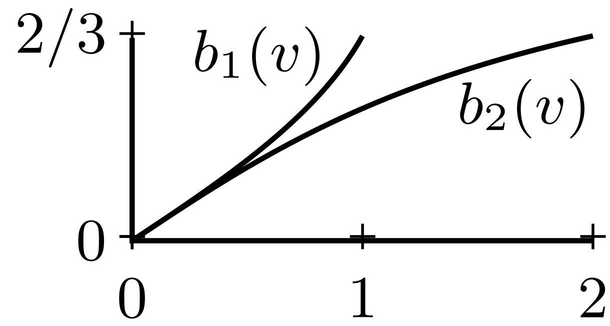

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Notes: solved by differential equation, 50 years to solve general uniform case, only for two bidders.

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PoA Analysis: quantify performance without solving for equilibrium.

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5. Sum over bidders, expectation over values:
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Proof by Picture:

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Proof by Picture:

Deviation Covering Lemma

Deviation Covering Lemma: $u_i(v_i, v_i/2) + \mathbf{E}[\text{BNE revenue}] \geq \frac{1}{2} v_i$

from bidder i (w. value v_i)	**from auction (and other bids)**
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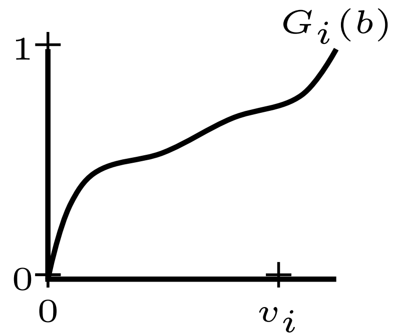

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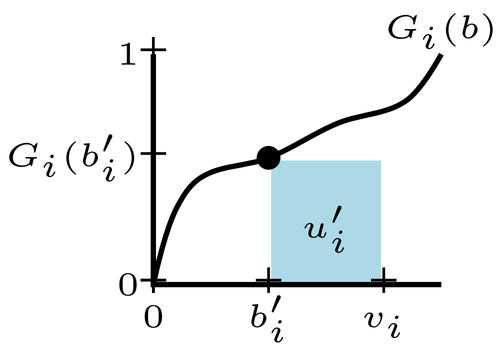

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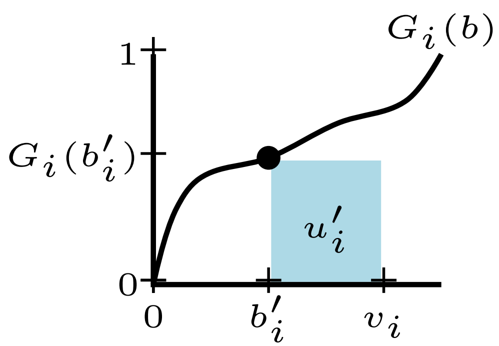

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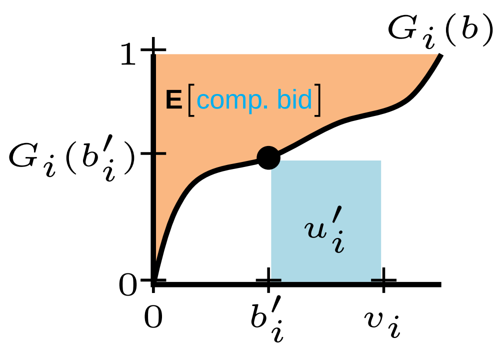

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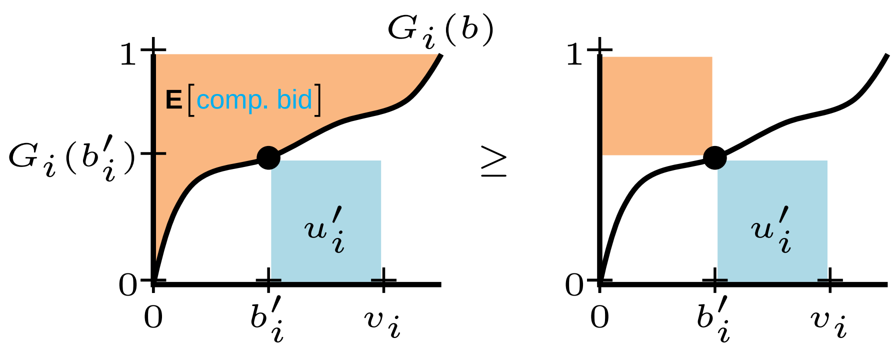

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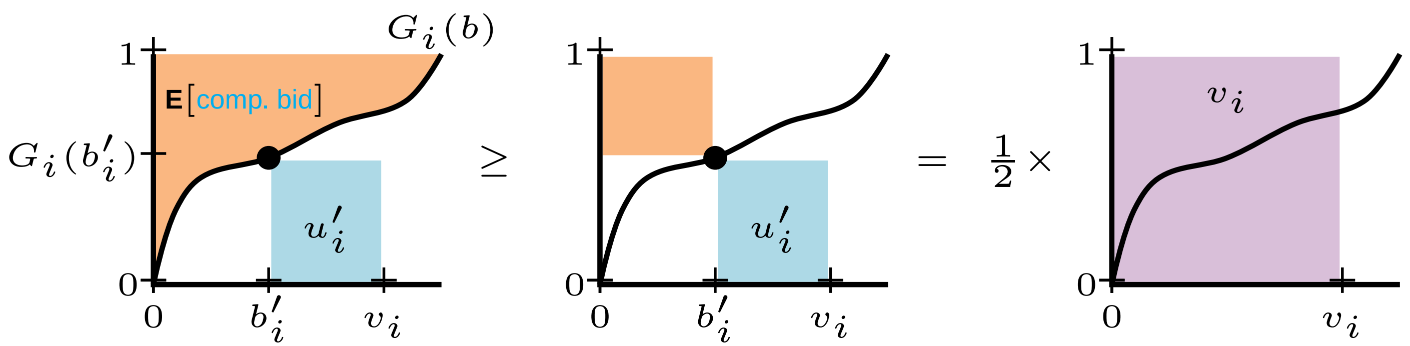

Questions?

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This tutorial: PoA for auctions (as games of incomplete information)

Overview of Tutorial

Part I: Introduction and motivation.

Part II: Smoothness Framework
(extension theorems, correlated dists., auction composition)

· · · coffee break · · ·

Part III: Standard Examples
(position auctions, multi-unit auctions, matching markets, combinatorial auctions)

Part IV: BNE Characterization and Consequences
(BNE characterization, symmetric BNE, solving, uniqueness, revenue)

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