ECON 2010c
Solution to Problem Set 1

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Fall 2014

1 Growth Model

(a) Defining the constant κ as:

$$\kappa = \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta),$$

the problem asks us to show that the following holds for each $n \geq 1$:

$$(T^n \hat{v})(k) = \frac{1-\beta^n}{1-\beta}\kappa + \hat{v}(k). \tag{1}$$

We can show this is true by induction where we first show that the statement holds for $n = 1$. Then, we will show that as long as the statement holds for some arbitrary $n-1$, it also holds for n.

For $n = 1$, the first application of the Bellman operator T to the starting function $\hat{v}$ yields:

$$(T\hat{v})(k) = \sup_{y\in[0,k^\alpha]} \{\ln(k^\alpha - y) + \beta\hat{v}\,(y)\} = \sup_{y\in[0,k^\alpha]} \left\{\ln(k^\alpha - y) + \frac{\alpha\beta}{1-\alpha\beta}\ln(y)\right\}. \tag{2}$$

Noting that the maximand is strictly concave on $[0, k^\alpha]$, there is a unique maximizer y^* that satisfies the first-order condition:

$$-\frac{1}{k^\alpha - y^*} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{y^*} = 0 \;\Rightarrow\; y^* = \alpha\beta k^\alpha,$$

Then, plugging this expression in for the optimal y in the maximand of (2) gives us:

$$\begin{aligned}(T\hat{v})(k) &= \ln(k^\alpha - \alpha\beta k^\alpha) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta k^\alpha) \\ &= \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \left[1 + \frac{\alpha\beta}{1-\alpha\beta}\right]\alpha\ln(k) = \kappa + \hat{v}(k).\end{aligned}$$

Thus, equation (1) is satisfied for $n = 1$. Assuming that $(T^{n-1}\hat{v})(k)$ is given by equation (1), $(T^n\hat{v})(k)$ can be calculated as follows:

$$\begin{aligned}(T^n\hat{v})(k) &= \sup_{y\in[0,k^\alpha]} \{\ln(k^\alpha - y) + \beta(T^{n-1}\hat{v})(y)\} \\ &= \sup_{y\in[0,k^\alpha]} \left\{\ln(k^\alpha - y) + \beta\left[\frac{1-\beta^{n-1}}{1-\beta}\kappa + \frac{\alpha\ln(y)}{1-\alpha\beta}\right]\right\}.\end{aligned}$$

Note that this is actually the same maximand as in (2) with an additional constant that doesn't depend on y and so the optimal y^* will remain the same as above, $y^* = \alpha\beta k^\alpha$. Plugging this in gives us

$$\begin{aligned}(T^n\hat{v})(k) &= \ln(k^\alpha - \alpha\beta k^\alpha) + \beta\left[\frac{1-\beta^{n-1}}{1-\beta}\kappa + \frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta k^\alpha)\right] \\ &= \left(\beta\frac{1-\beta^{n-1}}{1-\beta} + 1\right)\kappa + \left[1 + \frac{\alpha\beta}{1-\alpha\beta}\right]\alpha\ln(k) = \frac{1-\beta^n}{1-\beta}\kappa + \hat{v}(k).\end{aligned}$$

Hence, equation (1) holds for all $n \geq 1$ (in fact, you can clearly see that it also holds for $n = 0$). It follows that:

$$v(k) = \lim_{n\to\infty}(T^n\hat{v})(k) = \frac{1}{1-\beta}\kappa + \hat{v}(k).$$

Substituting this limit value function v into the right-hand side of the Bellman equation yields:

$$\begin{aligned}\sup_{y\in[0,k^\alpha]}\{\ln(k^\alpha - y) + \beta v(y)\} &= \frac{\beta}{1-\beta}\kappa + \sup_{y\in[0,k^\alpha]}\{\ln(k^\alpha - y) + \beta\hat{v}(y)\} = \frac{\beta}{1-\beta}\kappa + (T\hat{v})(k) \\ &= \frac{\beta}{1-\beta}\kappa + [\kappa + \hat{v}(k)] = v(k),\end{aligned}$$

confirming that v is indeed the solution to the Bellman equation.

(b) Substituting $v(k) = \psi + \phi\ln(k)$ into the Bellman equation and collecting the terms involving ψ on the LHS yields the following:

$$(1-\beta)\psi + \phi\ln(k) = \sup_{y\in[0,k^\alpha]}\{\ln(k^\alpha - y) + \beta\phi\ln(y)\}.$$

Noting that the maximand is strictly concave on $[0, k^\alpha]$, there is a unique maximizer y^* that satisfies the first-order condition:

$$-\frac{1}{k^\alpha - y^*} + \frac{\beta\phi}{y^*} = 0 \;\Rightarrow\; y^* = \frac{\beta\phi}{1+\beta\phi}k^\alpha.$$

It follows from the envelope theorem that the derivative of the Bellman equation with respect to k is:

$$\frac{\phi}{k} = \frac{\alpha k^{\alpha-1}}{k^\alpha - y^*} \;\Rightarrow\; y^* = \frac{\phi-\alpha}{\phi}k^\alpha.$$

Combining the first-order condition and the envelope condition gives:

$$\frac{\beta\phi}{1+\beta\phi} = \frac{\phi-\alpha}{\phi} \;\Rightarrow\; \phi = \frac{\alpha}{1-\alpha\beta}.$$

so that, we obtain the optimal policy $y^* = \alpha\beta k^\alpha$ after substituting for ϕ in the first-order condition. Next, substituting for ϕ and y in the Bellman equation yields:

$$\begin{aligned}
(1-\beta)\psi + \frac{\alpha}{1-\alpha\beta}\ln(k) &= \ln(k^\alpha - \alpha\beta k^\alpha) + \beta\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta k^\alpha) \\
&= \frac{\alpha}{1-\alpha\beta}\ln(k) + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) \\
\Rightarrow \psi &= \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right]
\end{aligned}$$

(c) The capital stock k_t^* evolves according to the recursive equation $k_t^* = \alpha\beta(k_{t-1}^*)^\alpha$ for each integer $t \geq 1$. The equation can be expressed in logs as:

$$\ln(k_t^*) = \ln(\alpha\beta) + \alpha\ln(k_{t-1}^*),$$

which is a first-order linear difference equation with some initial value $\ln k_0$. By backward induction, its (backward) solution is found to be

$$\begin{aligned}
\ln(k_t^*) &= \ln(\alpha\beta) + \alpha\left[\ln(\alpha\beta) + \alpha\ln(k_{t-2}^*)\right] \\
&= \ln(\alpha\beta) + \alpha\ln(\alpha\beta) + \alpha^2\left[\ln(\alpha\beta) + \alpha\ln(k_{t-3}^*)\right] \\
&= \ldots \\
&= \ln(\alpha\beta)\sum_{s=0}^{t-1}\alpha^s + \alpha^t\ln(k_0) \\
&= \frac{1-\alpha^t}{1-\alpha}\ln(\alpha\beta) + \alpha^t\ln(k_0).
\end{aligned}$$

Exponentiating both sides of the solution, we obtain:

$$k_t^* = (\alpha\beta)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t}.$$

Thus, the present discounted value of the flow payoffs from the sequence k_t^* can be calculated as follows:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \ln[(k_t^*)^\alpha - k_{t+1}^*] &= \sum_{t=0}^{\infty} \beta^t \ln[(k_t^*)^\alpha - \alpha\beta(k_t^*)^\alpha] = \sum_{t=0}^{\infty} \beta^t [\ln(1-\alpha\beta) + \alpha \ln(k_t^*)] \\
&= \sum_{t=0}^{\infty} \beta^t \left\{ \ln(1-\alpha\beta) + \alpha \left[\frac{1-\alpha^t}{1-\alpha} \ln(\alpha\beta) + \alpha^t \ln(k_0) \right] \right\} \\
&= \frac{1}{1-\beta} \ln(1-\alpha\beta) + \frac{\alpha}{1-\alpha} \ln(\alpha\beta) \sum_{t=0}^{\infty} \left[\beta^t - (\alpha\beta)^t \right] + \alpha \ln(k_0) \sum_{t=0}^{\infty} (\alpha\beta)^t \\
&= \frac{1}{1-\beta} \ln(1-\alpha\beta) + \frac{\alpha}{1-\alpha} \left(\frac{1}{1-\beta} - \frac{1}{1-\alpha\beta} \right) \ln(\alpha\beta) + \frac{\alpha}{1-\alpha\beta} \ln(k_0) \\
&= \frac{1}{1-\beta} \ln(1-\alpha\beta) + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)} \ln(\alpha\beta) + \frac{\alpha}{1-\alpha\beta} \ln(k_0) \\
&= \psi + \phi \ln(k_0) \\
&= v(k_0)
\end{aligned}$$

This confirms that the policy function $g(k) = \alpha\beta k^\alpha$ is indeed optimal since following this policy attains the optimal value that we found above for any initial k_0.

(d) The steady-state capital stock $\tilde{k}$ is the fixed point of the difference equation $k_t = \alpha\beta k_{t-1}^\alpha$. Hence, $\tilde{k}$ satisfies $\tilde{k} = \alpha\beta\tilde{k}^\alpha$ or, equivalently, $\alpha\beta\tilde{k}^{\alpha-1} = 1$. The first-order Taylor expansion of the policy function $k_{t+1} = \alpha\beta k_t^\alpha$ around $k_t = \tilde{k}$ yields:

$$k_{t+1} \approx \tilde{k} + \alpha^2\beta\tilde{k}^{\alpha-1}\left(k_t - \tilde{k}\right) \;\Rightarrow\; \frac{k_{t+1} - \tilde{k}}{k_t - \tilde{k}} \approx \alpha^2\beta\tilde{k}^{\alpha-1} = \alpha = e^{-[-\ln(\alpha)]}.$$

To see why α is the capital share (of output) in this model, note that the gross income/output derived from capital is k^α and thus, the marginal product of capital is $\alpha k^{\alpha-1}$. In a competitive market, the return rate r on a unit of capital is equal to this marginal product. Consequently, the share of output paid to capital is given by:

$$\frac{rk}{k^\alpha} = \frac{\alpha k^{\alpha-1} k}{k^\alpha} = \alpha.$$

A capital share α between 0.3 and 0.7 implies a convergence rate $-\ln(\alpha)$ at least as great as $-\ln(0.7) \approx 0.35$. One reason why the model fails to match the convergence rate of 0.05 in the data is that we have assumed that the depreciation rate of capital between periods is one. This assumption raises the convergence rate, because past investments in capital have a smaller effect on the current capital stock. Thus, an initial difference between the capital stocks of two economies declines at a more rapid rate. This intuition relies on an interpretation of the convergence rate as a measure of how quickly countries which started with different initial capital stocks will converge to the same steady state capital stock. This interpretation is valid to the extent that the different economies can be modeled with the same production technologies and preferences.

2 Equity Model

(a) The term $v(x)$ on the left-hand side of the Bellman equation represents the supremum of the present discounted value of flow payoffs that can be generated from a feasible consumption stream starting at a wealth level of x. On the right-hand side, the term $u(x-y)$ represents the current flow payoff from consuming the amount $x-y$, and the term

$$E\left[\exp(-\rho)v\left(\exp(r+\sigma u-\sigma^2/2)y\right)\right] = \exp(-\rho)E\left[v\left(\exp(r+\sigma u-\sigma^2/2)y\right)\right]$$

represents the present discounted value of the continuation payoff. The latter expression consists of the discount factor $\exp(-\rho)$ representing impatience and the expected value $E\left[v\left(\exp(r+\sigma u-\sigma^2/2)y\right)\right]$ of the optimization problem in the following period. Note that $\exp(r+\sigma u-\sigma^2/2)y$ is the wealth level in the following period given the realization of the return shock u. The current savings level y is chosen to maximize the value of the current flow payoff plus the the discounted expected future value of the problem, subject to the constraint that savings cannot be negative or greater than the current wealth level x.

(b) Given that $v(x) = \phi\frac{x^{1-\gamma}}{1-\gamma}$, the envelope theorem implies that:

$$v'(x) = u'(x-y) \Rightarrow \phi x^{-\gamma} = (x-y)^{-\gamma} = c^{-\gamma};$$

so that the optimal consumption function is $c = \phi^{-\frac{1}{\gamma}}x$ and the corresponding saving function is $y = (1-\phi^{-\frac{1}{\gamma}})x$. Denoting $R = \exp(r+\sigma u-\sigma^2/2)$, the first order condition is:

$$u'(x-y) = \exp(-\rho)E\left[Rv'(Ry)\right] \Rightarrow (x-y)^{-\gamma} = \exp(-\rho)E\left[R^{1-\gamma}\phi y^{-\gamma}\right]$$

Substituting the consumption and saving functions into the last expression and rearranging yields:

$$\phi x^{-\gamma} = \exp(-\rho)\phi(1-\phi^{-\frac{1}{\gamma}})^{-\gamma}x^{-\gamma}E\left[R^{1-\gamma}\right] \Rightarrow (1-\phi^{-\frac{1}{\gamma}})^{\gamma} = \exp(-\rho)E\left[R^{1-\gamma}\right].$$

Noting that R is a lognormal random variable (ie., $\ln(R)$ is normally distributed), the moment-generating function for the normal distribution implies that:

$$E\left[R^t\right] = E\left[e^{t\log R}\right] = \exp\left(tm + \frac{t^2}{2}s^2\right),$$

where m and s^2 are the mean and variance of $\ln(R)$. Since in our case, $m = r - \sigma^2/2$ and $s^2 = \sigma^2$, the preceding expression can be written as:

$$(1-\phi^{-\frac{1}{\gamma}})^{\gamma} = \exp\left[-\rho + (1-\gamma)\left(r - \frac{\sigma^2}{2}\right) + \frac{(1-\gamma)^2}{2}\sigma^2\right] = \exp\left[-\rho + (1-\gamma)r + \frac{(\gamma-1)\gamma}{2}\sigma^2\right]$$

so that, taking the log of both sides and dividing by γ yields:

$$\ln(1-\phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma}[(1-\gamma)r-\rho] + \frac{1}{2}(\gamma-1)\sigma^2.$$

(c) Using the consumption and saving functions derived above, we obtain that optimal consumption growth is given by:

$$\frac{c_{t+1}}{c_t} = \frac{\phi^{-\frac{1}{\gamma}}x_{t+1}}{\phi^{-\frac{1}{\gamma}}x_t} = R(1-\phi^{-\frac{1}{\gamma}})$$

where the last equality follows from the feasibility constraint

$$x_{+1} \;\; = \;\; Ry = R(1-\phi^{-\frac{1}{\gamma}})x$$

Then, we have:

$$\begin{aligned} E\ln\left(\frac{c_{t+1}}{c_t}\right) &= E\ln(R) + \ln(1-\phi^{-\frac{1}{\gamma}}) \\ &= \left(r-\frac{\sigma^2}{2}\right) + \left\{\frac{1}{\gamma}[(1-\gamma)r-\rho] + \frac{1}{2}(\gamma-1)\sigma^2\right\} = \frac{1}{\gamma}(r-\rho) + \left(\frac{\gamma}{2}-1\right)\sigma^2. \end{aligned}$$

(d) In the case where $\sigma = 0$, the Euler equation relating consumption across two periods can be expressed as $E\Delta\ln(c_{t+1}) = \frac{1}{\gamma}(r-\rho)$. A higher value of the interest rate r raises the price of current consumption relative to future consumption. In other words, current consumption becomes more costly because the opportunity cost represented by the expected return from investing in equity is higher. A higher value of the discount rate ρ means that a consumer is less patient. That is, the contribution of future utility flows to the discounted sum of utility flows becomes less important relative to current period utility. Hence, the growth rate $\Delta\ln(c_{t+1})$ of consumption is increasing in r and decreasing in ρ. The coefficient of relative risk aversion γ also regulates the consumer's willingness to substitute consumption between periods here because it measures the concavity of the flow utility function and thus the desire to smooth consumption across periods. Note that the elasticity of intertemporal substitution for this utility function is:

$$-\frac{d\ln\left(c_{t+1}/c_t\right)}{d\ln\left(u'\left(c_{t+1}\right)/u'(c_t)\right)} = \frac{1}{\gamma}$$

A higher value of γ corresponds to a more concave utility function and a lower willingness to substitute consumption across periods. Thus, the expected growth rate of consumption $E\Delta\ln(c_{t+1})$ is less sensitive to $r-\rho$.

(e) Because the amount of savings an agent wishes to allocate to the bond and the equity must be chosen optimally, the Euler equation must hold for each asset.

In other words, we are now choosing savings via both the equity and the bond. Thus, we can rewrite the Bellman as:

$$\tilde{v}(x) = \sup_{y_e+y_f\in[0,x]} u(x-y_e-y_f)+E\left\{\exp(-\rho)\,\tilde{v}\left[\exp\left(r+\sigma u-\sigma^2/2\right)y_e+\exp(r_f)\,y_f\right]\right\}$$

Then, it can be verified that the above optimal solution of $c=\phi^{-\frac{1}{\gamma}}x$ still satisfies the equity Euler equation we can get from combining the FOC w.r.t. y_e and the envelope condition. Now, the additional FOC w.r.t. y_f gives:

$$u'(c_t) = \exp(-\rho)E[\exp(r_f)\tilde{v}'(x_{t+1})]$$

where it is recognized that $c_t = x_t - y_{e,t} - y_{f,t}$ and $x_{t+1} = \exp(r+\sigma u - \sigma^2/2)\,y_{e,t} + \exp(r_f)\,y_{f,t}$. The envelope condition gives us $\tilde{v}'(x_{t+1}) = u'(x_{t+1} - y_{e,t+1} - y_{f,t+1})$. Thus, additionally using the budget constraint then gives that the interest rate r_f on the risk-free bond satisfies the bond Euler equation:

$$\begin{aligned} u'(c_t) &= \exp(-\rho)E[\exp(r_f)u'(c_{t+1})] \;\Rightarrow\; c_t^{-\gamma} = \exp(r_f-\rho)E(c_{t+1}^{-\gamma}) \\ &\Rightarrow\; r_f = \rho - \ln E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] \end{aligned}$$

Since the bond pays only an infinitesimal amount, the agent's optimal consumption path c_t will still be characterized by $\frac{c_{t+1}}{c_t} = R(1-\phi^{-\frac{1}{\gamma}})$, allowing us to express r_f as follows:

$$\begin{aligned} r_f &= \rho - \ln E\{[R(1-\phi^{-\frac{1}{\gamma}})]^{-\gamma}\} = \rho - \ln E(R^{-\gamma}) + \gamma\ln(1-\phi^{-\frac{1}{\gamma}}) \\ &= \rho - \left[-\gamma\left(r-\frac{\sigma^2}{2}\right)+\frac{\gamma^2\sigma^2}{2}\right] + \gamma\left\{\frac{1}{\gamma}[(1-\gamma)r-\rho]+\frac{1}{2}(\gamma-1)\sigma^2\right\} = r-\gamma\sigma^2. \end{aligned}$$

Note that the effective expected rate of return from equity is r_e which solves the equation

$$\exp(r_e) = E[R] = \exp(r) \Rightarrow r_e = r.$$

Then we have that, in equilibrium, this return must exceed the bond return in order to clear both the equity and bond markets:

$$r_e - r_f = \gamma\sigma^2.$$

This equity premium is a measure of the additional return (in terms of a real rate of return) that consumers are willing to forego in exchange for a sure payoff.

3 True/False/Uncertain

(a) True. Let $\Gamma(x)$ be nonempty for all $x \in X$ so that a feasible plan exists. In addition, assume that for every feasible plan $\{x_t\}_{t=0}^{\infty}$, the infinite sum $\sum_{t=0}^{\infty}\delta^t F(x_t, x_{t+1})$

exists (but can be plus or minus infinity) so that the value of every feasible plan can be evaluated using the objective function and thus our maximization problem is well-defined. Then, because every set of real numbers has a unique supremum in the extended real number system, every sequence problem must have a unique supremum value, holding fixed the initial state $x_0 \in X$.

(b) True. There are two alternative approaches to prove this. The first one makes use of Blackwell's sufficiency conditions and the Contraction Mapping Theorem. The second one makes use of the formulation equivalence results for sequential and recursive problems.

First Approach:

Consider the Bellman equation $v(x) = \max_{y\in\Gamma(x)}\{F(x,y) + \delta v(y)\}$ where F is a bounded function, and the discount factor satisfies $\delta \in (0,1)$. Then as long as we verify Blackwell's sufficiency conditions, we know that the corresponding Bellman operator is a contraction mapping. Furthermore, note that $B(X)$ (the space of bounded functions $f : X \to X$ with the sup-metric) is a complete metric space. Thus, by the contraction mapping theorem, there is a unique fixed point to the Bellman operator $T : B(X) \to B(X)$, that is, there exists a unique bounded function that solves the Bellman equation.

We now check that Blackwell's sufficient conditions are satisfied. For monotonicity, take any $f, g \in B(X)$ with $f \le g$. Then, the following holds for every $x \in X$:

$$\begin{aligned}(Tf)(x) &= \sup_{y\in\Gamma(x)}\{F(x,y) + \delta f(y)\} = F(x, y_f^*) + \delta f(y_f^*) \\ &\le F(x, y_f^*) + \delta g(y_f^*) \le \sup_{y\in\Gamma(x)}\{F(x,y) + \delta g(y)\} = (Tg)(x).\end{aligned}$$

For discounting, take $c \ge 0$ and $f \in B(X)$.Then, the following holds for every $x \in X$:

$$[T(f+c)](x) = \sup_{y\in\Gamma(x)}\{F(x,y) + \delta[f(y) + c]\} = \sup_{y\in\Gamma(x)}\{F(x,y) + \delta f(y)\} + \delta c = (Tf)(x) + \delta c.$$

Second Approach:

Suppose that the flow payoff function, $F(x,y)$, is bounded. Note that if B is a bound for $|F(x,y)|$, then the supremum (value) function V^{SP} of the sequence problem satisfies $|V^{SP}(x)| \le \frac{B}{1-\beta}$, for all $x \in X$, so that the latter is also bounded.

We can therefore prove the statement by making use of *both* parts of the main result establishing the (partial) equivalence of the sequence and recursive formulation, reproduced here:

Theorem 1 (Theorems 4.2 and 4.3 of Stokey and Lucas (1989))**.** *Suppose that the sequence problem satisfies the following two regularity conditions:*

- *$\Gamma(x)$ is nonempty for all $x \in X$*
- *For all feasible sequences, $\lim_{n\to\infty} \sum_{t=0}^{n} \delta^t F(x_t, x_{t+1})$ exists (but may be plus or minus infinity).*

Then,

(a) If V^{SP} is a solution to the sequence problem, then it is also a solution to the Bellman equation.

(b) If V^{BE} is a solution to the Bellman equation and $\lim_{t\to\infty} \delta^t V^{BE}(x_t) = 0$ for all feasible x_t, then it is also a solution to the sequence problem.

Armed with this theorem, we need to prove both existence and uniqueness of a bounded solution to the Bellman equation.

Proof of Existence of a Bounded Solution:

Given the two weak regularity conditions of Theorem 1 (which we have to assume), the solution V^{SP} to the sequence problem is also a solution to the Bellman equation by part (a) of the theorem. Since we showed above that V^{SP} is bounded, it follows that there exists at least one bounded solution to the Bellman equation.

Proof of Uniqueness of the Bounded Solution:

Suppose, for the sake of contradiction, that there are two or more distinct bounded solutions to the Bellman equation. Because every bounded solution to the Bellman equation satisfies $\lim_{t\to\infty} \delta^t V^{BE}(x_t) = 0$ for all feasible x_t, by part 2 of Theorem 1 we know that all such bounded solutions to the Bellman equation must also be solutions to the sequence problem. But this would contradict the uniqueness of V^{SP}, the supremum value of the sequence problem, discussed in question 3.a. above. Therefore, the bounded solution to the Bellman equation must be unique.

(c) False. *First counterexample:* Consider the Bellman equation

$$v(x) = \sup_{x_+ \in \Gamma(x)} F(x, x_+) + \beta \mathbb{E} V(x_+)$$

with $F(x, x_+) = x_+$ and $\Gamma(x) = Bx$ where $B \in (1, 1/\beta)$. Clearly, the optimal policy is trivially $x_+^* = Bx > x$ so that the flow payoff function grows unboundedly large under the optimal policy, $\lim_{n\to\infty} F(x_n^*, x_{n+1}^*) = \lim_{n\to\infty} x_{n+1}^* = \lim_{n\to\infty} B^n x_0 = \infty$.

But note that (by part (a) of Theorem 1 above) a solution $v(x)$ to this Bellman equation is the solution to the corresponding sequence problem,

$$\begin{aligned} v(x_0) &= \sup_{\{x_t\} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ &= x_0 + \beta B x_0 + \beta^2 B^2 x_0 + \dots \\ &= \frac{x_0}{1 - \beta B} < \infty \end{aligned}$$

where the last equality follows from the fact that $\beta B < 1$. Thus, for any x_n, we have $v(x_n) = x_n/(1 - \beta B)$ finite and $\lim_{n\to\infty} \beta^n v(x_n) = 0$.

Second counterexample: Consider the following optimal-stopping problem (example 3 from Section 1). There is a tree with initial size $x_0 > 0$ that grows by $\mu > 0$ units per period if left uncut. That is, we have $x_t = x_0 + t\mu$ for all $t \geq 0$. If a tree of size x is harvested, then the agent receives a payoff of x. The discount factor is $\delta \in (0, 1)$. The flow payoff from cutting a tree of size $x > 0$ is $F(x, 0) = x$, and the flow payoff in the periods before or after cutting the tree is $F(x, x + \mu) = F(0, 0) = 0$. Note that the flow payoff function $F(x, 0) = x$ is unbounded. The Bellman equation for this problem is:

$$v(x) = \max\{x, \delta v(x + \mu)\}.$$

Noting that $x^* = \delta\mu/(1 - \delta)$ is the threshold solution, we know that at least one solution $v(\cdot)$ of the Bellman equation is characterized by a policy rule where the tree is harvested iff $x \geq x^*$. Thus, we know that $v(x) = x$ for $x > x^*$ and $v(x) \leq x^*$ for $x \leq x^*$. We also see that for any x_0, $x_t = x_0 + t\mu \geq x^*$ for some finite t. Thus, for t sufficiently large, we have $x_t > x^*$ and so $v(x_t) \leq x_t = x_0 + t\mu$. Noting also that the flow payoff is always nonnegative, the following holds for any feasible sequence of actions:

$$0 \leq \lim_{t\to\infty} \delta^t v(x_t) \leq \lim_{t\to\infty} \delta^t (x_0 + t\mu) = 0 \;\Rightarrow\; \lim_{t\to\infty} \delta^t v(x_t) = 0.$$

(d) True. The maximum feasible capital stock $\bar{k}_t$ at time t satisfies $\bar{k}_t = \bar{k}_{t-1}^{\alpha}$ or $\ln(\bar{k}_t) = a^t \ln(k_0)$. Since $|a| < 1$, we have $\lim_{t\to\infty} \ln(\bar{k}_t) = 0$, so that $\lim_{t\to\infty} k_t \leq \lim_{t\to\infty} \bar{k}_t = 1 < 1 + \varepsilon$.

(e) True. From the form of the solution $v(\cdot)$ found in the first problem, we know that v is monotonically increasing in its argument. We also know that $k_t \leq \bar{k}_t = k_0^{\alpha^t}$ from the preceding result regarding the maximum feasible capital stock. Thus,

$$\lim_{t\to\infty} \beta^t v(k_t) \leq \lim_{t\to\infty} \beta^t v(\bar{k}_t) = \lim_{t\to\infty} \beta^t [\phi + \psi\alpha^t \ln(k_0)] = 0, \text{ since } |\alpha\beta| < 1.$$

Takeaways from Problem 1 and the True/False questions

The growth model of Problem 1 and the T/F questions above illustrate that some of the most important economic applications of dynamic programming involve cases where the boundedness condition

$$\lim_{t\to\infty} \beta^t v(x_t) = 0 \quad \text{for all feasible } \{x_t\} \tag{3}$$

does *not* hold. This condition is a *sufficient* one for ensuring that the solution to the Bellman equation is also the solution to the sequence problem. A variation of this equivalence result that applies in more general circumstances is the following:

Lemma 1 (Exercise 4.3 in Stokey and Lucas (1989)). *If $v(x)$ is a solution to the Bellman equations that satisfies*

$$\limsup_{t\to\infty} \beta^t v(x_t) \leq 0 \tag{4}$$

for all feasible sequences $\{x_t\}_{t=0}^{\infty}$ and, in addition, an optimal feasible sequence $\{x_t^*\}_{t=0}^{\infty}$ for the sequence problem exists and satisfies

$$\lim_{t\to\infty} \beta^t v(x_t^*) = 0 \tag{5}$$

then $v(x)$ is also the solution to the corresponding sequence problem.

As we have seen in 3.e, the first condition of Lemma 1 is satisfied, while in Problem 1.d we showed that the optimal sequence under our solution v to the Bellman equation converges to the steady-state level of capital, $\tilde{k} = (\alpha\beta)^{\frac{1}{1-\alpha}}$, so that[1]

$$\begin{aligned}
\lim_{t\to\infty} \beta^t v(k_t^*) &= \lim_{t\to\infty} \beta^t \left[\psi + \phi \ln(k_t^*)\right] \\
&= \lim_{t\to\infty} \beta^t \left[\psi + \phi \ln(\tilde{k})\right] \\
&= 0
\end{aligned}$$

Therefore, we know that the solution $v(k)$ to the Bellman equation of the growth model is also the solution to the sequence formulation of the growth model. Moreover, by part 3.a., we know that the solution to the latter is unique, so that we have computed the unique solution to the growth model.

There is another useful equivalence result that, unlike Lemma 1, that does not require that we have computed an optimal sequence for the sequence problem, $\{x_t^*\}$. This theorem only requires that the Bellman operator is monotonic (and not necessarily a contraction). The idea is to start with a function $\hat{v}$ that is an upper bound for v^{SP} and then apply the operator T to $\hat{v}$, iterating down to a fixed point.

Theorem 2 (Theorem 4.14 in Stokey and Lucas (1989)). *Define Bellman operator T on the set of* all *functions $f : X \to \mathbb{R}$ by*

$$(Tf)(x) = \sup_{y\in\Gamma(x)} \left[F(x,y) + \beta f(y)\right]$$

Assume that T is monotone, that is, $f \leq g \Rightarrow Tf \leq Tg$.

Moreover, assume that there is a function $\hat{v} : X \to \mathbb{R}$ such that

1. $T\hat{v} \leq \hat{v}$

2. $\lim_{t\to\infty} \beta^t \hat{v}(x_t) \leq 0$ *for all feasible x_t*

[1] Strictly speaking, we need to show that this is the optimal sequence of capital levels for the sequence problem, and not just under the candidate solution v to the Bellman equation.

3. $V^{SP}(x_0) \leq \hat{v}(x_0)$ *for all* $x_0 \in X$

If the function $v : X \to \bar{\mathbb{R}}$ *defined by*

$$v = \lim_{n\to\infty} (T^n \hat{v})(x)$$

is a fixed point of T*, then* $v = v^{SP}$*.*

In Problem 1.a. of Problem Set 1, we showed that the growth model with the upper bound guess $\hat{v}(k) = \frac{a \ln k}{1-\alpha\beta}$ satisfies the assumptions of Theorem 2. Therefore, we know that $v(k) = \lim_{n\to\infty}(T^n\hat{v})(k)$ is not only a solution to the Bellman equation (that is, a fixed point of T, a fact that we showed by simply plugging the expression for $v(k)$ back into the Bellman equation) but also the solution to the sequence problem.

Relatedly, many economic problems feature payoff functions that are unbounded, so that we cannot make use of Blackwell's theorem to show that the Bellman operator is a contraction and then apply the contraction mapping theorem to prove the existence of a unique solution to the Bellman Equation, as we did in 3.b under the assumption of a bounded function. However, note that if our payoff function is continuous and we can, without loss of generality, restrict our attention to a *compact* subset of the state space (equivalently, a closed and bounded subset of the state space when the latter is a Euclidean space, $\mathbb{R}^n$, for some finite $n \in \mathbb{N}$), then our payoff function (restricted to the compact subset of the spate space) will be bounded, so that we can apply the contraction mapping theorem.

The growth model of Problem 1 is such a case. From part 3.d., we see that the economy can never settle into a level of capital greater than the value defined by $\bar{k} = \bar{k}^\alpha \Rightarrow \bar{k} = 1$, since this is the amount of capital that would sustain itself when consumption is set to 0. If the economy starts with $k(0) < 1$, it can never exceed 1. If it starts with $k(0) > 1$, it can never exceed $k(0)$. Then, we can take $\hat{k} \equiv \max\{k(0), 1\}$ and *equivalently* formulate our problem as one where the agent's choice for next period's capital is formally restricted to be in the compact set $[0, \hat{k}^\alpha - \varepsilon]$, for some $\varepsilon > 0$ sufficiently small.[2]

[2] In this particular example we need consumption to be strictly positive, because the flow payoff function, log utility of consumption, tends to $-\infty$ as consumption approaches zero, so that the flow payoff remains unbounded even if the choice for next period's capital is restricted to $[0, \hat{k}^\alpha]$. The restriction that next period's capital cannot exceed $\hat{k}^\alpha - \varepsilon$, for small enough $\varepsilon > 0$, can again be shown to be without loss of generality.