ECON 2010c
Solution to Problem Set 3

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1 A Simple Consumption Problem

(a) Noting that $v_T(x) = \ln x$, the Bellman equation for period $T-1$ is:

$$v_{T-1}(x) = \sup_{c\in[0,x]} \{\ln c + \beta \ln[R(x-c)]\},$$

whose first-order condition is:

$$\frac{1}{c} = \frac{\beta}{x-c},$$

yielding the policy rule:

$$c_{T-1}(x) = \frac{x}{1+\beta} = \lambda_{T-1}x.$$

Thus, the value function for period $T-1$ is:

$$v_{T-1}(x) = \ln\left(\frac{x}{1+\beta}\right) + \beta\ln\left(\frac{\beta R x}{1+\beta}\right) = (1+\beta)\ln\left(\frac{x}{1+\beta}\right) + \beta\ln(R\beta).$$

The Bellman equation for period $T-2$ is:

$$v_{T-2}(x) = \sup_{c\in[0,x]} \{\ln c + \beta v_{T-1}[R(x-c)]\} = \sup_{c\in[0,x]} \left\{\ln c + \beta(1+\beta)\ln\left[\frac{R(x-c)}{1+\beta}\right] + \beta^2\ln(R\beta)\right\},$$

whose first-order condition is:

$$\frac{1}{c} = \frac{\beta(1+\beta)}{x-c},$$

yielding the policy rule:

$$c_{T-2}(x) = \frac{x}{1+\beta+\beta^2} = \lambda_{T-2}x.$$

Thus, the value function for period $T-2$ is:

$$\begin{aligned} v_{T-2}(x) &= \ln\left(\frac{x}{1+\beta+\beta^2}\right) + \beta(1+\beta)\ln\left(\frac{R\beta x}{1+\beta+\beta^2}\right) + \beta^2\ln(R\beta) \\ &= (1+\beta+\beta^2)\ln\left(\frac{x}{1+\beta+\beta^2}\right) + (\beta+2\beta^2)\ln(R\beta). \end{aligned}$$

(b) It will be shown by induction that the policy rule is as claimed in the problem set and that the value function is given by:

$$v_{T-t}(x) = \frac{\ln(\lambda_{T-t}x)}{\lambda_{T-t}} + \kappa_{T-t}\ln(R\beta),$$

where we define:

$$\kappa_{T-t} = \sum_{s=0}^{t} s\beta^s = \frac{\beta}{(1-\beta)^2}\left\{1-\beta^t\left[1+t\,(1-\beta)\right]\right\}.$$

It is easily seen that the policy rule and the value function are as claimed above for $t=0$. If these claims are true for some $t \geq 0$, then the Bellman equation for $t+1$ is:

$$v_{T-t-1}(x) = \sup_{c\in[0,x]}\{\ln c + \beta v_{T-t}[R(x-c)]\} = \sup_{c\in[0,x]}\left\{\ln c + \beta\left[\frac{\ln[\lambda_{T-t}R(x-c)]}{\lambda_{T-t}} + \kappa_{T-t}\ln(R\beta)\right]\right\},$$

whose first-order condition is:

$$\frac{1}{c} = \frac{\beta}{\lambda_{T-t}(x-c)},$$

yielding the policy rule for $t+1$:

$$c_{T-t-1}(x) = \frac{\lambda_{T-t}}{\beta+\lambda_{T-t}}x = \frac{(1-\beta)(1-\beta^{t+1})^{-1}}{\beta+(1-\beta)(1-\beta^{t+1})^{-1}}x = \frac{1-\beta}{1-\beta^{t+2}}x = \lambda_{T-t-1}x.$$

Thus, the value function for $t+1$ is:

$$\begin{aligned} v_{T-t-1}(x) &= \ln(\lambda_{T-t-1}x) + \frac{\beta}{\lambda_{T-t}}\ln[\lambda_{T-t}R(1-\lambda_{T-t-1})x] + \beta\kappa_{T-t}\ln(R\beta) \\ &= \left(1+\frac{\beta}{\lambda_{T-t}}\right)\ln(\lambda_{T-t-1}x) + \frac{\beta}{\lambda_{T-t}}\ln\left[\frac{\lambda_{T-t}(1-\lambda_{T-t-1})}{\beta\lambda_{T-t-1}}\right] \\ &\quad +\beta\left(\frac{1}{\lambda_{T-t}}+\kappa_{T-t}\right)\ln(R\beta) \\ &= \frac{\ln(\lambda_{T-t-1}x)}{\lambda_{T-t-1}} + \frac{\beta}{\lambda_{T-t}}\ln\left\{\frac{1-\lambda_{T-t}\left(\beta+\lambda_{T-t}\right)^{-1}}{\beta\left(\beta+\lambda_{T-t}\right)^{-1}}\right\} + \beta\left(\sum_{s=0}^{t}\beta^s + \sum_{s=0}^{t}s\beta^s\right)\ln(R\beta) \\ &= \frac{\ln(\lambda_{T-t-1}x)}{\lambda_{T-t-1}} + \frac{\beta}{\lambda_{T-t}}\ln\left\{\frac{[(\beta+\lambda_{T-t})-\lambda_{T-t}]}{\beta}\right\} + \kappa_{T-t-1}\ln(R\beta) \\ &= \frac{\ln(\lambda_{T-t-1}x)}{\lambda_{T-t-1}} + \kappa_{T-t-1}\ln(R\beta), \end{aligned}$$

which confirms the claim.

(c) The limiting policy rule is:

$$c(x) = \lim_{t\to\infty} c_{T-t}(x) = \lim_{t\to\infty} \lambda_{T-t} x = (1-\beta)x,$$

and the limiting value function is:

$$v(x) = \lim_{t\to\infty} v_{T-t}(x) = \lim_{t\to\infty} \left\{ \frac{\ln(\lambda_{T-t}x)}{\lambda_{T-t}} + \kappa_{T-t} \ln(R\beta) \right\} = \frac{\ln[(1-\beta)x]}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(R\beta).$$

(d) Letting $\lambda = 1-\beta$ and $\kappa = \beta/(1-\beta)^2$, we confirm that:

$$v(x) = \frac{\ln(\lambda x)}{\lambda} + \kappa \ln(R\beta)$$

is the solution to the Bellman equation:

$$v(x) = \sup_{c\in[0,x]} \{\ln c + \beta v[R(x-c)]\}.$$

Substituting for $v(x)$ yields:

$$\lambda^{-1}\ln(\lambda x) + \kappa \ln(R\beta) = \sup_{c\in[0,x]} \left\{ \ln c + \beta \frac{\ln[\lambda R(x-c)]}{\lambda} + \beta\kappa \ln(R\beta) \right\}.$$

The first-order condition is given by:

$$\frac{1}{c} = \frac{\beta}{\lambda(x-c)} \Rightarrow c = \frac{\lambda}{\beta+\lambda} x;$$

so that, we obtain the following:

$$\lambda^{-1}\ln(\lambda x) + \kappa \ln(R\beta) = \frac{\beta+\lambda}{\lambda} \ln\left(\frac{\lambda x}{\beta+\lambda}\right) + \beta(\frac{1}{\lambda} + \kappa) \ln(R\beta),$$

which is satisfied for λ and κ defined above.

(e) The policy rule falls with each iteration, indicating that the marginal propensity to consume decreases as the agent inducts backwards from the end of the problem. The reason for this change is that the agent has a lesser incentive to save for future periods as she approaches the end of the problem.

The value function becomes steeper with each iteration:

$$V'_{T-t}(x) = \frac{\sum_{s=0}^{t} \beta^s}{x}$$

which is increasing in t. This is because at the earlier stages of the problem, the agent has a greater ability to smooth consumption across periods, thereby mitigating the effect of diminishing marginal utility from consumption.

Note that whether the *level* of $v_{T-t}(x)$ increases or falls with t depends on the specific values of R and β, as well as x.

2 True/False/Uncertain

1. Uncertain. On the one hand, as the simple consumption problem of problem 1 illustrates, a simple life-cycle model where agents are not liquidity constrained might predict that 55-year-olds have a higher marginal propensity to consume than 25-year-olds, because of the shorter remaining lifespan of 55-year-olds. This argument would suggest that the statement is false. On the other hand, income profiles tend to be increasing over a worker's lifetime, and capital markets are unlikely to be perfect; so that, many 25-year-olds could be liquidity constrained. Hence, 25-year-olds might consume the entire tax cut, while 55-year-olds would save much of it, in order to smooth consumption over the rest of their lives. This argument would suggest that the statement is true.

2. Uncertain. In a simple model with perfect credit markets, the life-cycle hypothesis would indicate that consumption should be constant over time, provided that $\delta R = 1$, where R is constant and nonrandom.[1] Even more generally, if the 15% wage increase in 2014 was previously expected (prior to 2013), then consumption in 2013 would not be further affected by the wage increase. That is, consumption plans would be revised when the wage increase was initially announced. If the 15 percent wage increase in 2014 was not expected before 2013, then consumption should rise in 2013, as long as borrowing is possible. In the presence of liquidity constraints, however, the worker might not be able to finance higher consumption in 2013.

3. True. If we interpret the term "binding" to mean that the consumer is spending all of their cash on hand each period ($c_t = x_t$), then for small (marginal) changes in their cash on hand, they will continue to spend everything, which implies a marginal propensity to consume equal to one. For large enough tax rebates, the liquidity constraint might cease to bind, and the MPC would be less than one.

3 Three-Period Hyperbolic Discounting Model

(a) The utility function at $t = 2$ is a linear transformation of the last two terms of the utility function at $t = 1$ iff there exists $\kappa \in \mathbb{R}$ such that for all $(c_2, c_3) \in \mathbb{R}^2_{++}$:

$$\ln(c_2) + \beta\delta \ln(c_3) = \kappa[\beta\delta \ln(c_2) + \beta\delta^2 \ln(c_3)].$$

This condition is satisfied iff there exists $\kappa \in \mathbb{R}$ such that $\kappa\beta\delta = 1$ and $\kappa\beta\delta^2 = \beta\delta$ or, equivalently, $\kappa = (\beta\delta)^{-1}$ and $\kappa = \delta^{-1}$. If $\beta = 1$, then the condition is satisfied for $\kappa = \delta^{-1}$. If $\beta < 1$, then no $\kappa \in \mathbb{R}$ satisfies the condition.

[1] Even if $\delta R \neq 1$ the entire consumption path would be predetermined and not affected at all by the wage increase in the perfect capital markets case since a worker would have optimally sold all claims to his future labor income stream at the beginning of his working life.

In order to show that selves one and two do not have the same rankings over points in (c_2, c_3)-space if $\beta < 1$, note that self one is indifferent between the points (e, e) and $(e^{1+\delta}, 1)$ and that self two is indifferent between the points (e, e) and $(e^{1+\beta\delta}, 1)$. Thus, for any $\varepsilon \in (\beta\delta, \delta)$, self one prefers (e, e) to $(e^{1+\varepsilon}, 1)$ and self two prefers $(e^{1+\varepsilon}, 1)$ to (e, e).

(b) If commitment is possible, then self one faces the optimization problem:

$$\max_{c_1, c_2, c_3} \{u(c_1) + \beta u(c_2) + \beta u(c_3)\},$$

subject to: $c_1 + c_2 + c_3 \le A_1$. Noting that $u(\cdot) = \ln(\cdot)$ is increasing and concave with $u'(0) = \infty$, so the unique solution to this problem satisfies the first-order condition:

$$u'(c_1) = \beta u'(c_2) = \beta u'(c_3) \;\Rightarrow\; c_1^{-1} = \beta c_2^{-1} = \beta c_3^{-1} \;\Rightarrow\; \beta c_1 = c_2 = c_3.$$

(c) In the case where $\beta < 1$, self two places a higher weight than self one on utility received in period two relative to utility received in period three. Thus, self two has an incentive to revise self one's consumption program, so as to increase consumption in period two and decrease consumption in period three. Self two faces the optimization problem:

$$\max_{c_2, c_3} \{u(c_2) + \beta u(c_3)\},$$

subject to: $c_2 + c_3 \le A_2$. The solution satisfies the first-order condition:

$$u'(c_2) = \beta u'(c_3) \;\Rightarrow\; \beta c_2 = c_3.$$

(d) Because u is strictly increasing, self three's unique solution is to consume all remaining assets and thus $c_3 = A_3 = A_2 - c_2$. Consequently, self two faces the optimization problem in the previous part, whose first-order condition $u'(c_2) = \beta u'(c_3)$ yields $\beta c_2 = c_3$. Hence, the solution to self two's problem is:

$$c_2 = \frac{c_3}{\beta} = \frac{A_2 - c_2}{\beta} \Rightarrow c_2 = \frac{A_2}{1+\beta} \;\text{ and }\; c_3 = \frac{\beta A_2}{1+\beta}.$$

Given the strategies of selves two and three, self one faces the optimization problem:

$$\max_{c_1} \left\{ \ln(c_1) + \beta \ln\left(\frac{A_2}{1+\beta}\right) + \beta \ln\left[\frac{\beta A_2}{1+\beta}\right] \right\} \quad \text{s.t. } c_1 + A_2 \le A_1,$$

which is equivalent to solving:

$$\max_{c_1} \{\ln(c_1) + 2\beta \ln(A_1 - c_1)\}.$$

The first-order condition yields:

$$\frac{1}{c_1} = \frac{2\beta}{A_1 - c_1};$$

so that, the consumption path is:

$$c_1 = \frac{A_1}{1+2\beta}, \; c_2 = \frac{A_1 - c_1}{1+\beta} = \frac{2\beta A_1}{(1+\beta)(1+2\beta)}, \text{ and } c_3 = \frac{\beta\,(A_1 - c_1)}{1+\beta} = \frac{2\beta^2 A_1}{(1+\beta)(1+2\beta)}.$$

(e) For log utility and $\delta = R = 1$, the generalized Euler equation simplifies to:

$$\frac{1}{c_t} = \left[1 - (1-\beta)\frac{\partial c_{t+1}}{\partial A_{t+1}}\right]\frac{1}{c_{t+1}}.$$

The solution from the previous part yields $\partial c_2/\partial A_2 = (1+\beta)^{-1}$ and $\partial c_3/\partial A_3 = 1$; so that, the generalized Euler equation requires:

$$\frac{c_2}{c_1} = 1 - \frac{1-\beta}{1+\beta} = \frac{2\beta}{1+\beta} \;\text{ and }\; \frac{c_3}{c_2} = 1 - (1-\beta) = \beta,$$

both of which are satisfied by the solution from the previous part.

The hyperbolic Euler equation differs from the standard Euler equation for an exponential discounter in that the discount factor δ is replaced by a weighted average of the short-term discount factor $\beta\delta$ and the long-term discount factor δ. The weight on $\beta\delta$ is next period's marginal propensity to consume $\partial c_{t+1}/\partial A_{t+1}$, and the weight on δ is equal to $(1 - \partial c_{t+1}/\partial A_{t+1})$.

The standard perturbation argument is invalid for a (sophisticated) hyperbolic discounter, because the current self does not actually choose future consumption. In particular, self one would prefer to consume less in period one and more in period three if self two could be prevented from revising the consumption path selected by self one. The next part shows that a perturbation exists that increases the utility of each of the three selves.

(f) Let $(\widehat{c}_1, \widehat{c}_2, \widehat{c}_3)$ represent the equilibrium path of consumption, and consider the perturbed consumption path $(\widehat{c}_1 - \Delta, \widehat{c}_2, \widehat{c}_3 + \Delta)$ for small $\Delta > 0$. Clearly, selves two and three prefer the perturbed path to the equilibrium path since consumption is weakly higher in periods 2 and 3. Self one prefers the perturbed path iff:

$$\begin{aligned} u(\widehat{c}_1 - \Delta) + \beta u(\widehat{c}_2) + \beta u(\widehat{c}_3 + \Delta) &> u(\widehat{c}_1) + \beta u(\widehat{c}_2) + \beta u(\widehat{c}_3) \\ \Leftrightarrow \beta[u(\widehat{c}_3 + \Delta) - u(\widehat{c}_3)] &> u(\widehat{c}_1) - u(\widehat{c}_1 - \Delta). \end{aligned}$$

For small $\Delta > 0$, a first-order Taylor approximation can be used to rewrite the last condition as:

$$\beta u'(\widehat{c}_3)\Delta > u'(\widehat{c}_1)\Delta \Leftrightarrow u'(\widehat{c}_1)/u'(\widehat{c}_3) < \beta \Leftrightarrow \widehat{c}_3/\widehat{c}_1 < \beta.$$

The equilibrium path found above satisfies this condition since:

$$\frac{\widehat{c}_3}{\widehat{c}_1} = \frac{2\beta^2}{1+\beta} < \beta,$$

where the inequality follows because $\beta \in (0, 1)$. Thus, there does exist a $\Delta > 0$ such that the perturbed consumption path Pareto-dominates the equilibrium path.

(g) In order to show that any consumption path $(c_1, c_2, c_3) \in \mathbb{R}^3_{++}$ satisfying the budget constraint $c_1 + c_2 + c_3 = A_1$ is a Nash equilibrium, consider the following strategy profile: (1) self one consumes c_1 regardless of the strategies of self two and three; (2) self two consumes c_2 if self one consumes c_1, and 0 otherwise; (3) self three consumes $A_1 - c_1 - c_2$ if selves one and two consume c_1 and c_2, and 0 otherwise. On the equilibrium path, self three consumes all the remaining wealth and so has no incentive to deviate. Selves one and two do not have an incentive to deviate, because any deviation would result in a utility of $-\infty$. Thus, the preceding strategy profile is a Nash equilibrium of the game.

Nonetheless, this strategy profile does not constitute a subgame-perfect equilibrium, because it relies on the non-credible threat of zero consumption off the equilibrium path. The subgame-perfect equilibria of this finite game of perfect information can be obtained through backward induction; so that, the unique backward-induction consumption path is also the unique subgame-perfect consumption path.

In an infinite-horizon version of the model, the folk theorem suggests that a large set of feasible consumption paths might be supported as subgame-perfect equilibria by appropriately defining finite punishment phases.

4 A Procrastination Problem

(a) Suppose that all selves follow the proposed strategy. Once the late fee for the current period has been sunk, each self decides between the loss c from completing the task and the loss $\beta(1+c)$ from postponing the task. However, the assumption $c > \beta(1 + c)$ implies that each self has an incentive to deviate by postponing the task. Thus, the proposed strategy cannot be an equilibrium.

(b) Suppose that all selves follow the proposed strategy. Once the late fee for the current period has been sunk, each self decides between the loss c from completing the task and the loss ∞ from postponing the task. Thus, each self has an incentive to deviate by completing the task; so that, the proposed strategy cannot be an equilibrium.

(c) Once the late fee for the current period has been sunk, the loss to the current self from completing the task after s periods is c for $s = 0$ and $\beta(s + c)$ for $s > 0$. Noting that $\beta(s + c)$ is increasing in s and that we've assumed $c > \beta(1 + c)$, this implies that the loss is minimized for $s = 1$. Thus, self one would commit to finish the task in period two.

(d) A naïve agent falsely believes that she can commit to finishing the task in the following period. Thus, a naïve agent would never finish the task and would receive a payoff of $-\infty$.

(e) The continuation loss V for an agent facing an unfinished task is the sum of: the late fee 1 for the current period; the cost c of completing the task times the probability p of completing the task in the current period; and the continuation loss V times the probability $(1 - p)$ of postponing the task. The agent bases her decision on the current loss function W, which is the minimum of the loss $(1+c)$ from completing the task in the current period and the loss $(1 + \beta V)$ from postponing the task. If $p = 1$, then it must be that $c \leq \beta V$, in order for the agent to be willing to always complete the task with probability 1. If $0 < p < 1$, then it must be that $c = \beta V$, so that the agent is exactly indifferent between completing and postponing the task. (Note that that for $p = 0$ the relationship need only hold as an inequality, $c \geq \beta V$; this is the mirror case of $p = 1$.)

(f) For $p > 0$, the expression for the continuation value function V can be rearranged to yield:

$$V = \frac{1 + pc}{p}.$$

Note that we can't have $p = 1$ since this gives us $V = 1+c$ and by assumption, $c > \beta(1 - c) = \beta V$ which means there would be a (strict) incentive for a sophisticated agent to postpone the task, as we saw in part a. Moreover, it cannot be the case that $p = 0$ if $\beta > 0$, because $p = 0$ implies that $V = \infty$, providing the agent with a strict incentive to always complete the task with probability 1. This final point corresponds to the case we saw in part b. Thus, we know that the equilibrium must be characterized by $p \in (0, 1)$ and so it must be the case that $c = \beta V$:

$$\frac{c}{\beta} = V = \frac{1 + pc}{p} \Rightarrow p = \frac{\beta}{(1 - \beta)c}.$$

Note that the resulting value of p is always strictly less than one as long as we have the assumption that $c > \beta(1 + c)$ and p will be strictly greater than 0 as long as $\beta > 0$. Thus, we will always have $\min\left\{\frac{\beta}{(1-\beta)c}, 1\right\} = \frac{\beta}{(1-\beta)c} = p$.

(g) As p approaches one, the task is completed in the current period with probability arbitrarily close to one; thus, the loss tends to$1 + c$ (which is the loss realized when the task is completed in the current period with probability 1).

(h) As β approaches one, the agent does not discount the cost of completing the task in the future; so that, it is more preferable to complete the task earlier to avoid the future late fees.

(i) In equilibrium, there is a probability $p = \beta/[(1-\beta)c]$ of completing an unfinished task in the current period (ie., in one period). Thus, the expected completion time T follows a geometric distribution with probability mass function for $n \geq 1$:

$$\Pr(T = n) = p(1-p)^{n-1}$$

and thus expectation:

$$E(T) = \frac{1}{p} = \frac{1-\beta}{\beta}c,$$

which is greater than one, given the assumption that $c > \beta(1+c)$. The expected completion time serves as a measure of procrastination because an agent with a higher β, indicating preferences closer to dynamic consistency, will complete the project with less delay. Note that in the dynamically consistent case ($\beta = 1$), we'd have to drop the $c > \beta(1+c)$ assumption since there won't exist a c which satisfies this assumption for $\beta = 1$. In that case, since the agent doesn't discount, the unique equilibrium would be characterized by $p = 1$ and so the project is completed in the first period with probability 1.