

ECON 2010c

Solution to Problem Set 4

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1 Consumption-Based Capital Asset Pricing Model

1. Substituting:

$$\delta = \exp(-\rho), \quad R_{t+1}^i = \exp[r_{t+1}^i + \sigma^i \varepsilon_{t+1}^i - \frac{1}{2}(\sigma^i)^2], \quad u'(c) = c^{-\gamma}$$

into the Euler equation:

$$u'(c_t) = E_t[\delta R_{t+1}^i u'(c_{t+1})]$$

yields:

$$c_t^{-\gamma} = E_t\{\exp[-\rho + r_{t+1}^i + \sigma^i \varepsilon_{t+1}^i - \frac{1}{2}(\sigma^i)^2] c_{t+1}^{-\gamma}\} \Rightarrow 1 = E_t\{\exp[-\rho + r_{t+1}^i + \sigma^i \varepsilon_{t+1}^i - \frac{1}{2}(\sigma^i)^2] (c_{t+1}/c_t)^{-\gamma}\}.$$

Noting that:

$$\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \exp\left\{\ln\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]\right\} = \exp\left[-\gamma \ln\left(\frac{c_{t+1}}{c_t}\right)\right] = \exp[-\gamma \Delta \ln(c_{t+1})],$$

the Euler equation can be written as:

$$1 = E_t\{\exp[-\rho + r_{t+1}^i + \sigma^i \varepsilon_{t+1}^i - \frac{1}{2}(\sigma^i)^2 - \gamma \Delta \ln(c_{t+1})]\}.$$

2. The moment-generating function for a normal random variable X is:

$$E[e^{bX}] = \exp(b\mu + \frac{1}{2}b^2\sigma^2),$$

where μ and σ^2 respectively denote the mean and variance of X . Applying this formula to the normal random variable $[\sigma^i \varepsilon_{t+1}^i - \gamma \Delta \ln(c_{t+1})]$ yields:

$$E_t\{\exp[\sigma^i \varepsilon_{t+1}^i - \gamma \Delta \ln(c_{t+1})]\} = \exp\{-\gamma E_t[\Delta \ln(c_{t+1})] + \frac{1}{2} V_t[\sigma^i \varepsilon_{t+1}^i - \gamma \Delta \ln(c_{t+1})]\}.$$

Hence, the Euler equation can be expressed as:

$$\begin{aligned} 1 &= \exp\{-\rho + r_{t+1}^i - \frac{1}{2}(\sigma^i)^2 - \gamma E_t[\Delta \ln(c_{t+1})] + \frac{1}{2}V_t[\sigma^i \varepsilon_{t+1}^i - \gamma \Delta \ln(c_{t+1})]\} \\ \Rightarrow 0 &= -\rho + r_{t+1}^i - \frac{1}{2}(\sigma^i)^2 - \gamma E_t[\Delta \ln(c_{t+1})] + \frac{1}{2}V_t[\sigma^i \varepsilon_{t+1}^i - \gamma \Delta \ln(c_{t+1})]. \end{aligned}$$

The Euler equation for the risk-free asset with $\sigma^f = 0$ is simply:

$$0 = -\rho + r_{t+1}^f - \gamma E_t[\Delta \ln(c_{t+1})] + \frac{1}{2}V_t[\gamma \Delta \ln(c_{t+1})].$$

Differencing the Euler equation for assets i and j yields:

$$\begin{aligned} 0 &= r_{t+1}^i - r_{t+1}^j - \frac{1}{2}[(\sigma^i)^2 - (\sigma^j)^2] + \frac{1}{2}\{V_t[\sigma^i \varepsilon_{t+1}^i - \gamma \Delta \ln(c_{t+1})] - V_t[\sigma^j \varepsilon_{t+1}^j - \gamma \Delta \ln(c_{t+1})]\} \\ \Rightarrow r_{t+1}^i - r_{t+1}^j &= \frac{1}{2}\{(\sigma^i)^2 - (\sigma^j)^2 - V_t[\sigma^i \varepsilon_{t+1}^i - \gamma \Delta \ln(c_{t+1})] + V_t[\sigma^j \varepsilon_{t+1}^j - \gamma \Delta \ln(c_{t+1})]\}. \end{aligned}$$

3. Using the definitions:

$$V(A) = E\{[A - E(A)]^2\}, \quad Cov(A, B) = E\{[A - E(A)][B - E(B)]\},$$

we can express:

$$\begin{aligned} V(A + B) &= E\{[(A + B) - E(A + B)]^2\} = E\{[A - E(A) + B - E(B)][A - E(A) + B - E(B)]\} \\ &= E\{[A - E(A)]^2 + 2[A - E(A)][B - E(B)] + [B - E(B)]^2\} \\ &= E\{[A - E(A)]^2\} + 2E\{[A - E(A)][B - E(B)]\} + E\{[B - E(B)]^2\} \\ &= V(A) + 2Cov(A, B) + V(B). \end{aligned}$$

4. The statistical lemma yields:

$$V_t[\sigma^k \varepsilon_{t+1}^k - \gamma \Delta \ln(c_{t+1})] = (\sigma^k)^2 - 2\gamma \sigma_{kc} + \gamma^2 V_t[\Delta \ln(c_{t+1})];$$

so that, we obtain:

$$\begin{aligned} \pi_{t+1}^{ij} &= r_{t+1}^i - r_{t+1}^j \\ &= \frac{1}{2}\{(\sigma^i)^2 - (\sigma^j)^2 - [(\sigma^i)^2 - 2\gamma \sigma_{ic} + \gamma^2 V_t[\Delta \ln(c_{t+1})]] + [(\sigma^j)^2 - 2\gamma \sigma_{jc} + \gamma^2 V_t[\Delta \ln(c_{t+1})]]\} \\ &= \gamma(\sigma_{ic} - \sigma_{jc}). \end{aligned}$$

5. Because $\sigma_{fc} = 0$, we have $\pi_{t+1}^{if} = \gamma \sigma_{ic} \forall i$. The equation states that the risk premium π_{t+1}^{if} on asset i is equal to the coefficient of relative risk aversion γ times the covariance σ_{ic} of asset returns with consumption growth. Intuitively, if the covariance σ_{ic} of asset returns with consumption growth is high, then the asset tends to deliver higher returns in states where the future marginal utility of consumption is low relative to today's marginal utility. It is therefore perceived to be a risky asset and a high risk premium π_{t+1}^{if} is needed to ensure that the consumer is willing to hold it. This effect becomes larger as the coefficient of relative risk aversion γ increases, because the more risk averse the agent the higher

the return that the agent requires as compensation for a given “quantity” of risk (the latter is captured by the covariance term). In other words, γ corresponds to the “price” of risk.

Note: The γ multiplying the asset’s covariance with (contemporaneous) consumption growth in the equity premium expression represents risk preferences rather than consumption smoothing preferences. This cannot be discerned from the derivation above but can be shown to be true by looking at the Epstein-Zin utility case (see section 3 in section note 3 for a brief discussion of the relationship between CRRA and Epstein-Zin utility).

2 Asset Pricing and the Equity Model

1. The Euler equation sets the marginal loss in utility from saving an additional unit of wealth in the current period equal to the marginal gain in expected discounted utility from consuming the resulting savings in the next period. If the Euler equation did not hold for some asset, then the consumer would have an incentive either to invest more in that asset or to invest less in the asset and possibly sell it short. Hence, the Euler equation needs to hold for every asset in equilibrium.
2. By definition, the Euler equation for the risk-free asset is:

$$u'(c_t) = E_t[R^f \exp(-\rho)u'(c_{t+1})].$$

This can be derived (along with a corresponding equity Euler equation) from the first-order and envelope conditions from a Bellman equation where the savings technology is through both an equity and risk-free asset. There, the consumer would optimally choose both current consumption and allocation of wealth across the two assets.

3. Because R^f is a constant, rearrangement of the Euler equation yields:

$$(R^f)^{-1} = E_t[\exp(-\rho)u'(c_{t+1})/u'(c_t)] = E_t[\exp\{-\rho + \ln[u'(c_{t+1})/u'(c_t)]\}].$$

For the utility function $u(c) = c^{1-\gamma}/(1-\gamma)$, we can express:

$$\ln \left[\frac{u'(c_{t+1})}{u'(c_t)} \right] = \ln \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] = -\gamma \Delta \ln(c_{t+1}).$$

Thus, we have the result:

$$(R^f)^{-1} = E_t\{\exp[-\rho - \gamma \Delta \ln(c_{t+1})]\}.$$

4. Substituting $c_t = \phi^{-\frac{1}{\gamma}}x_t$ into the equation of motion for x_t yields:

$$x_{t+1} = R_{t+1}^{equity}(x_t - c_t) = R_{t+1}^{equity}(1 - \phi^{-\frac{1}{\gamma}})x_t \Rightarrow x_{t+1}/x_t = R_{t+1}^{equity}(1 - \phi^{-\frac{1}{\gamma}}).$$

Hence, we can express $\Delta \ln(c_{t+1})$ as:

$$\begin{aligned}\Delta \ln(c_{t+1}) &= \ln(c_{t+1}/c_t) = \ln(x_{t+1}/x_t) = \ln(1 - \phi^{-\frac{1}{\gamma}}) + \ln(R_{t+1}^{equity}) \\ &= \left\{ \frac{1}{\gamma}[(1 - \gamma)r - \rho] + \frac{1}{2}(\gamma - 1)\sigma^2 \right\} + \left\{ r + \sigma\varepsilon_{t+1} - \frac{1}{2}\sigma^2 \right\} \\ &= \frac{1}{\gamma}(r - \rho) + \frac{\gamma}{2}\sigma^2 - \sigma^2 + \sigma\varepsilon_{t+1},\end{aligned}$$

where ε_{t+1} is a standard normal random variable. Thus, we have:

$$E_t[\Delta \ln(c_{t+1})] = \frac{1}{\gamma}(r - \rho) + \frac{\gamma}{2}\sigma^2 - \sigma^2 \quad \& \quad V_t(\Delta \ln(c_{t+1})) = V_t(\sigma\varepsilon_{t+1}) = \sigma^2$$

5. Using the moment-generating function for the normal distribution, the result from the third part can be rewritten as follows:

$$\begin{aligned}(R^f)^{-1} &= \exp\{-\rho - \gamma E_t[\Delta \ln(c_t)] + \frac{1}{2}\gamma^2 V_t[\Delta \ln(c_{t+1})]\} \\ &= \exp\{-\rho - \gamma[\frac{1}{\gamma}(r - \rho) + (\frac{\gamma}{2} - 1)\sigma^2] + \frac{1}{2}\gamma^2 \sigma^2\} = \exp(-r + \gamma\sigma^2).\end{aligned}$$

Taking the log of both sides yields the result: $r^f = r - \gamma\sigma^2$. In the previous problem, we obtained the Consumption Capital Asset Pricing Equation:

$$\pi^{if} = r^i - r^f = \gamma\sigma_{ic} = \gamma Cov_t(\sigma^i \varepsilon_{t+1}^i, \Delta \ln(c_{t+1}));$$

so that, we have the following in the current problem:

$$r^{equity} - r^f = \gamma Cov_t[\sigma\varepsilon_{t+1}, \ln(1 - \phi^{-\frac{1}{\gamma}}) + \ln(R_{t+1}^{equity})] = \gamma Cov_t(\sigma\varepsilon_{t+1}, \sigma\varepsilon_{t+1}) = \gamma V_t(\sigma\varepsilon_{t+1}) = \gamma\sigma^2.$$

6. It is realistic to assume that physical production technologies are, at least to some extent, risky. Therefore, it is plausible to assume that the aggregate supply of the riskfree asset is zero.

Note that this is perfectly consistent with the trading of riskfree claims in financial markets. Of course, in a closed economy with a single (representative) agent, the risk-free rate in general equilibrium must be such that the agent is indifferent between buying and selling the risk-free bond, because a representative agent cannot borrow from or lend to herself.

Also note that “net supply” is not the same as “excess supply” in this context. For example, equity is in positive net supply (there is positive aggregate endowment of equity) but its excess supply must be zero in equilibrium, as for any other asset.

7. It is not true in the real world that the variance of equity returns is equal to the covariance of equity returns with consumption growth. First, consumers hold assets other than equities, such as real estate, commodities, and cash. Second, consumers also receive labor income, which creates a noncontractible source of risk.

3 General Exam Question

1. To derive the stated Euler equation, we need to use the two defining conditions of (general, competitive) equilibrium: first, all (price-taking) agents behave optimally given the prices they face; second, markets clear. Let α_{tree} and α_{firm} be the units of the tree asset and the firm that the representative agent holds. The second condition, market clearing (demand must equal supply for all assets), requires that $\alpha_{\text{tree}} = 1$ and $\alpha_{\text{firm}} = 1$.

The first condition requires that this equilibrium portfolio allocation is optimal for the representative agent. That is, risk premia for each asset must be such that the representative agent is willing to hold exactly the net supply of each asset. It may be more intuitive to think of the representative agent as a continuum of identical agents of total mass equal to one.¹ Each infinitesimal agent takes prices as given and chooses the optimal portfolio allocation given these prices. Optimality then requires that each agent has no incentive to (marginally) perturb his holdings of each asset in a feasible way starting from the equilibrium holdings $\alpha_{\text{tree}} = 1$ and $\alpha_{\text{firm}} = 1$.

Consider the gain from marginally increasing the agent's holding of the tree by $d\alpha_{\text{tree}}$ units. This perturbation must respect the budget constraint if it is to be feasible. Note that, since p denotes the price of the firm and c is the price of the tree (by the fact that one certain unit of consumption in period 1 is worth 1 and the Law of One Price), the agent must reduce his holdings of the tree by $|d\alpha_{\text{firm}}| = \frac{c}{p}d\alpha_{\text{tree}}$ units so as to pay for the increase in tree holdings.

His consumption will now be $c + 1 + c \cdot d\alpha_{\text{tree}} - 1 \cdot |d\alpha_{\text{firm}}|$ in the good state and $c + c \cdot d\alpha_{\text{tree}}$ in the bad state.

Optimality requires that $U_{\text{new}} - U_{\text{old}} = 0$ for a marginal change in asset holdings. Taking a first-order expansion of U_{new} around the old portfolio allocation, we get

$$\begin{aligned} \mu [u(c + 1 + c \cdot d\alpha_{\text{tree}} - 1 \cdot |d\alpha_{\text{firm}}|) - u(c + 1)] + (1 - \mu) [u(c + c \cdot d\alpha_{\text{tree}}) - u(c)] &= 0 \\ \mu \left(c \cdot d\alpha_{\text{tree}} - 1 \cdot \frac{c}{p}d\alpha_{\text{tree}} \right) u'(c + 1) + (1 - \mu) (c \cdot d\alpha_{\text{tree}}) u'(c) &= 0 \\ (1 - p)\mu u'(c + 1) - p(1 - \mu)u'(c) &= 0 \end{aligned}$$

where the final line follows by multiplying the second line by $-\frac{p}{cd\alpha_{\text{tree}}}$.

An alternative, equivalent way of deriving this equation is to consider feasible perturbations of consumption across states of nature rather than perturbations in asset holdings. This can be done by looking at the prices of the Arrow-Debreu (AD) securities corresponding to each state of nature. In general, the AD security

¹Each agent also has an identical endowment of each asset, which aggregates to the net supply of each asset.

for state s is defined to be a financial claim paying 1 unit of consumption in state s and 0 in all other states. Thus, the price of the state- s AD security can be interpreted as the price of consumption in state s . For this reason, the prices of AD securities are known as *state prices*.

In our case, we have two states, a “good” state occurring with probability μ and a “bad” state occurring with probability $1 - \mu$. By the Law of One Price (LOOP), the state price for the good state is p , since the firm asset has an identical payoff schedule (that is, identical payoff in each possible state of nature) to that of the good-state AD security. Also note that, again by the LOOP, state prices must sum to 1, the price of one certain unit of consumption in period 1. Thus, the price of the bad-state AD security is $1 - p$.

The state prices pin down the feasible reallocations of consumption across the two states. For example, increasing consumption in the bad state by dc_{bad} units costs $(1 - p)dc_{\text{bad}}$, which must be offset through a sale of $|dc_{\text{good}}| = \frac{1-p}{p}dc_{\text{bad}}$ units of consumption in the good state. Optimality then requires

$$-\frac{1-p}{p}dc_{\text{bad}}\mu u'(c+1) + dc_{\text{bad}}(1-\mu)u'(c) = 0$$

Multiplying by $-\frac{p}{dc_{\text{bad}}}$ once again yields the Euler equation.

2. Denote $\hat{p} = \lim_{c \rightarrow \infty} p$. Rearranging the formula above and taking limits yields:

$$\frac{(1-\hat{p})\mu}{\hat{p}(1-\mu)} = \lim_{c \rightarrow \infty} \frac{u'(c)}{u'(c+1)} = \lim_{c \rightarrow \infty} \frac{c+1}{c} = 1;$$

so that, we have:

$$(1-\hat{p})\mu = \hat{p}(1-\mu) \Leftrightarrow \hat{p} \equiv \lim_{c \rightarrow \infty} p = \mu.$$

In terms of the expected gross “return”, we have $E_0(\text{payout}) = \mu \cdot 1 + (1-\mu) \cdot 0 = \mu$. Thus,

$$\lim_{c \rightarrow \infty} \frac{E_0(\text{payout})}{p} = \lim_{c \rightarrow \infty} \frac{\mu}{p} = 1$$

3. In the limit as c goes to infinity, the ratio of consumption in the good state to consumption in the bad state approaches one. Given the CRRA form of the agent’s utility function, the consumer’s marginal utility is approximately equal across states because consumption becomes approximately equal across states *in proportional terms*. Therefore, if the price of a unit of the numeraire delivered in both states is one, then the price of the firm is simply equal to the probability that it pays a dividend of a unit of the numeraire (or its expected payout). Intuitively, as $c \rightarrow \infty$ the CRRA agent perceives no risk, so that risk aversion becomes irrelevant.

- 4/5. We are considering the same perturbation as in part 1. The difference is that the equilibrium payout that the consumer receives in the good state is now $(1 + \alpha)c + 1$ instead of $c + 1$. Again, suppose that the consumer sells a claim on one (marginal) unit of consumption delivered in the good state (that is, he sells a unit of the good-state AD security) and buys a claim on $p(1 - p)^{-1}$ units of consumption delivered in the bad state (that is, he buys $p(1 - p)^{-1}$ units of the bad-state AD security). This perturbation again satisfies the consumer's budget constraint with equality. Noting that markets clear in equilibrium, the loss in expected utility from one unit less of the numeraire in the good state is $\mu u'[(1 + \alpha)c + 1]$, and the gain in expected utility from $p(1 - p)^{-1}$ units more of the numeraire in the bad state is $p(1 - p)^{-1}(1 - \mu)u'(c)$. Because the consumer is maximizing her utility in equilibrium, she must be indifferent towards this perturbation; therefore, we obtain:

$$-\mu u'[(1 + \alpha)c + 1] + p(1 - p)^{-1}(1 - \mu)u'(c) = 0 \Leftrightarrow (1 - p)\mu u'[(1 + \alpha)c + 1] - p(1 - \mu)u'(c) = 0.$$

6. Denote $\tilde{p} = \lim_{c \rightarrow \infty} p$. Rearranging the formula above and taking limits yields:

$$\frac{(1 - \tilde{p})\mu}{\tilde{p}(1 - \mu)} = \lim_{c \rightarrow \infty} \frac{u'(c)}{u'[(1 + \alpha)c + 1]} = 1 + \alpha;$$

so that, we have:

$$(1 - \tilde{p})\mu = \tilde{p}(1 - \mu)(1 + \alpha) \Leftrightarrow \tilde{p} \equiv \lim_{c \rightarrow \infty} p = \frac{\mu}{1 + \alpha(1 - \mu)}$$

Similarly, we have

$$\lim_{c \rightarrow \infty} \frac{E_0(\text{payout})}{p} = \lim_{c \rightarrow \infty} \frac{\mu}{p} = 1 + \alpha(1 - \mu) > 1$$

7. Even in the limit as c goes to infinity, the ratio of consumption in the good state to consumption in the bad state is greater than one. Given that the consumer has CRRA preferences, this implies that marginal utility is greater in the bad state than in the good state. Because the firm pays a dividend when marginal utility is relatively low, the firm is perceived to be a risky investment, so that its price is less than the expected dividend (it commands a risk premium). Risk aversion is important in the current setting because consumption differs across states in proportional terms, whereas the ratio of consumption in the two states approached one in the previous case.

This exercise highlights the key idea of modern asset pricing theory: the riskiness of an asset or gamble does not depend on its volatility but on its comovement with other risks that agents face in equilibrium (equivalently, comovement with marginal utility or with the stochastic discount factor).

8. The calibration is inconsistent with the empirical fact that equity returns are weakly correlated with consumption growth. In the calibration, consumption is always high in the state where the firm has a high return and low in the state where the firm has a low return; that is, the correlation between equity returns and consumption growth is assumed to be one. In the postwar period, the correlation is estimated at around 0.18 for the US and it is even lower for other countries (for some, it is even negative).

The calibration is also inconsistent with the low observed volatility of consumption growth. The given calibration implies that the unconditional standard deviation of $\frac{c_{t+1}-c_t}{c_t} \approx \Delta \ln c_{t+1}$ is approximately 0.08. The estimated standard deviation of US consumption growth in the postwar period is quite lower, at around 0.0164 (and similarly low for other countries).

9. The equity premium is negative if $\alpha < 1$, because the firm now pays a dividend in the state where consumption is lower and the marginal utility of consumption is higher. In other words, firm equity serves a hedging role and, therefore, the agent is willing to accept a negative equity premium to hold equity.