ECON 2010c
Solution to Problem Set 5

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1 Merton's Consumption Problem

- The Bellman equation for the problem is given by:
$$\rho V(x)dt = \max_{c,\theta}\{u(c)dt + E(dV)\},$$

where the equation of motion for x is $dx = [(r + \theta\pi)x - c]dt + \theta\sigma x dz$. Using Ito's lemma, the Bellman equation can be expressed as:
$$\rho V(x) = \max_{c,\theta}\{u(c) + [(r + \theta\pi)x - c]V' + \tfrac{1}{2}(\theta\sigma x)^2 V''\}.$$

Substituting for $V(x)$ and $u(c)$ yields:
$$\rho[\phi + \psi\log(x)] = \max_{c,\theta}\{\log(c) + [(r + \theta\pi)x - c]\psi x^{-1} - \tfrac{1}{2}(\theta\sigma x)^2\psi x^{-2}\}$$
$$\Rightarrow\ \rho[\phi + \psi\log(x)] = \psi r + \max_{c,\theta}\{\log(c) - \psi c x^{-1} + \psi\pi\theta - \tfrac{1}{2}\psi\sigma^2\theta^2\}.$$

The first-order conditions for c and θ yield:
$$c^{-1} - \psi x^{-1} = 0\ \Rightarrow\ c = \frac{1}{\psi}x,$$
$$\psi\pi - \psi\theta\sigma^2 = 0\ \Rightarrow\ \theta = \frac{\pi}{\sigma^2}.$$

- Substituting these results into the Bellman equation gives:
$$\rho\phi + \rho\psi\log(x) = \psi r - \log(\psi) - 1 + \frac{\psi\pi^2}{2\sigma^2} + \log(x).$$

The Bellman equation is satisfied $\forall x$ iff:
$$\rho\phi = \psi r - \log(\psi) - 1 + \frac{\psi\pi^2}{2\sigma^2}\ \text{ and }\ \rho\psi = 1,$$

whence we obtain:
$$\psi = \frac{1}{\rho},\ \ \phi = \frac{1}{\rho}\left(\frac{r}{\rho} + \log(\rho) - 1 + \frac{\pi^2}{2\rho\sigma^2}\right).$$

Thus, the optimal policy is $c = \rho x$ and $\theta = \pi/\sigma^2$.

2 True/False/Uncertain

1. False. The claim does not hold for an arbitrary Ito process $x(t)$, although it is true for a Wiener process $z(t)$. For example, if $x(t)$ follows the geometric Brownian motion $dx = \alpha x dt + \sigma x dz$, then the variance of $x(t) - x(0)$ is given by:
$$V[x(t) - x(0)] = [x(0)]^2 e^{2\alpha t}(e^{\sigma^2 t} - 1),$$
which is not a linear function of t.

2. False. An example of a mean-reverting Ito process $x(t)$ is the Ornstein-Uhlenbeck process $dx = \eta(\mu - x)dt + \sigma dz$, whose expected value is given by:
$$E[x(t)] = \mu + [x(0) - \mu]\exp^{-\eta t},$$
which converges to μ as t becomes large.

3. False. A Wiener process, and thus an Ito process, is everywhere continuous but nowhere differentiable, almost surely.

3 Optimal Stopping with Arithmetic Brownian Motion

- Note the typo: the γ's in the threshold rule expression should be ρ's. The expression for the optimal threshold can be rewritten as follows
$$x^* = -\frac{b^2}{a + \sqrt{a^2 + 2b^2\rho}} - \frac{a}{\rho} = -\frac{b^2\left(\sqrt{a^2 + 2b^2\rho} - a\right)}{\left(\sqrt{a^2 + 2b^2\rho} + a\right)\left(\sqrt{a^2 + 2b^2\rho} - a\right)} - \frac{a}{\rho}$$
$$= -\frac{1}{\rho}\left(\frac{\sqrt{a^2 + 2b^2\rho} - a}{2} + a\right) = -\frac{\sqrt{a^2 + 2b^2\rho} + a}{2\rho} < 0.$$

- Evaluating the expression for x^* at $a = 0$ yields:
$$x^*_{(a=0)} = -\frac{b}{\sqrt{2\rho}} < 0.$$

In the absence of drift, the firm stays in business longer if market volatility is higher or if the discount rate is lower. Intuitively, the option value of staying in business increases with market volatility and decreases with the discount rate.

- It is clear that:
$$\lim_{a \to \infty} x^* = -\infty.$$

As the drift becomes infinitely positive, the expected profit from remaining in business goes to positive infinity; so that, the firm never exits the market.

- As the drift becomes infinitely negative, we have:

$$\lim_{a \to -\infty} x^* = -\lim_{a \to \infty} \frac{-a + \sqrt{a^2 + 2b^2\rho}}{2\rho} = 0;$$

so that, the firm exits the market as soon as current profits become negative. An infinitely negative drift prevents the firm from earning positive profits once current profits become negative.

- Evaluating the expression for x^* at $b = 0$ yields:

$$\lim_{b \to 0} x^* = -\frac{a + |a|}{2\rho} = \begin{cases} -\frac{a}{\rho}, & a > 0 \\ 0, & a \le 0 \end{cases}.$$

In the absence of uncertainty, the firm stays in business iff the present discounted value of future profits is positive.

- It is clear that:
$$\lim_{b \to \infty} x^* = -\infty.$$

As market volatility becomes infinitely large, the option value of staying in business becomes infinitely positive; so that, the firm never exits the market.

- The threshold x^* can be rewritten as:

$$x^* = -\frac{\left(\sqrt{a^2 + 2b^2\rho} + a\right)\left(\sqrt{a^2 + 2b^2\rho} - a\right)}{2\rho\left(\sqrt{a^2 + 2b^2\rho} - a\right)} = -\frac{b^2}{\sqrt{a^2 + 2b^2\rho} - a};$$

hence, we obtain:
$$\lim_{\rho \to 0} x^* = -\frac{b^2}{|a| - a} = \begin{cases} -\infty, & a \ge 0 \\ \frac{b^2}{2a}, & a < 0 \end{cases}.$$

If the firm does not discount the future and there is a positive drift, then the present value of future profits is infinitely positive; thus, the firm never exits the market. In the case of a negative drift, the length of time that the firm stays in business increases with the volatility of the market, because greater volatility raises the option value of waiting.

- It is clear that:
$$\lim_{\rho \to \infty} x^* = 0.$$

If the firm does not value future profits, then the firm exits the market iff current profits are negative.

4   Optimal Stopping with Geometric Brownian Motion

- In order to ensure that x^* is finite and our problem well-defined, assume that $\rho > a$. The Bellman equation for the continuation region where $x \geq x^*$ is given by:

$$\rho V(x)dt = w(x)dt + E(dV).$$

  The flow payoff function is $w(x) = x - c$, and Ito's lemma provides $E(dV) = (axV' + \frac{1}{2}b^2x^2V'')dt$. Thus, the Bellman equation for $x \geq x^*$ can be expressed as:

$$\rho V(x) = x - c + axV'(x) + \tfrac{1}{2}b^2x^2V''(x) \Rightarrow \tfrac{1}{2}b^2x^2V''(x) + axV'(x) - \rho V(x) + (x - c) = 0,$$

  which provides a differential equation for V. The value-matching and smooth-pasting conditions yield $V(x^*) = \Omega(x^*) = 0$ and $V'(x^*) = \Omega'(x^*) = 0$.

- Next, consider the value $\widetilde{V}(x)$ of a policy of never exiting the industry. It is possible to compute $\widetilde{V}$ as follows:

$$\widetilde{V}[x(0)] = E\left\{\int_0^\infty e^{-\rho t}[x(t) - c]dt\right\} = \int_0^\infty e^{-\rho t}E[x(t)]dt - c\int_0^\infty e^{-\rho t}dt$$

$$= x(0)\int_0^\infty e^{-\rho t}e^{at}dt - c\int_0^\infty e^{-\rho t}dt = \frac{x(0)}{\rho - a} - \frac{c}{\rho},$$

  where the third equality follows from the fact that $E[x(t)] = x(0)e^{at}$ for a geometric Brownian motion $x(t)$ with drift a. It is easily confirmed that:

$$\rho\widetilde{V}(x) = \rho\left[\frac{x}{\rho - a} - \frac{c}{\rho}\right] = x - c + \frac{ax}{\rho - a} = x - c + ax\widetilde{V}'(x) + \tfrac{1}{2}b^2x^2\widetilde{V}''(x);$$

  so that, $\widetilde{V}$ is a particular solution to the differential equation for V. In order to obtain a solution to the homogeneous equation, consider a solution of the form $g(x) = Cx^\gamma$. Substituting into the differential equation gives:

$$\tfrac{1}{2}b^2\gamma(\gamma - 1)Cx^\gamma + a\gamma Cx^\gamma - \rho Cx^\gamma = 0 \Rightarrow \tfrac{1}{2}b^2\gamma^2 + \left(a - \tfrac{1}{2}b^2\right)\gamma - \rho = 0$$

$$\Rightarrow \gamma = \frac{1}{2} - \frac{a}{b^2} \pm \sqrt{\left(\frac{a}{b^2} - \frac{1}{2}\right)^2 + 2\frac{\rho}{b^2}}$$

  Letting γ^+ and γ^- respectively denote the positive and negative roots of the quadratic equation, the general solution to the differential equation is given by:

$$V(x) = \widetilde{V}(x) + C^- x^{\gamma^-} + C^+ x^{\gamma^+}.$$

  As the state variable x becomes infinitely positive, the option value of never exiting the industry approaches zero; so that, $V(x)$ converges to $\widetilde{V}(x)$. In particular,

$\lim_{x\to\infty} V'(x) = \lim_{x\to\infty} \widetilde{V}'(x) = \frac{1}{\rho-a}$. Because $\gamma^+ > 1$ and $\gamma^- < 0$, it must be that $C^+ = 0$; otherwise, $V'(x)$ would be unbounded as x goes to ∞. The smooth-pasting condition yields:

$$V'(x^*) = \frac{1}{\rho - a} + \gamma^- C^- (x^*)^{\gamma^- - 1} = 0 \Rightarrow C^- = -\frac{(x^*)^{1-\gamma^-}}{(\rho - a)\gamma^-}.$$

Substituting this result into the value-matching condition gives:

$$V(x^*) = \frac{x^*}{\rho - a} - \frac{c}{\rho} + C^-(x^*)^{\gamma^-} = \frac{x^*}{\rho - a} - \frac{c}{\rho} - \frac{x^*}{(\rho - a)\gamma^-} = 0 \Rightarrow x^* = \frac{\rho - a}{\rho}\frac{\gamma^-}{\gamma^- - 1}c.$$

Thus, the final solution for the constant C^- is given by:

$$C^- = -\frac{1}{(\rho - a)\gamma^-}\left(\frac{\rho - a}{\rho}\frac{\gamma^-}{\gamma^- - 1}c\right)^{1-\gamma^-} = \frac{c^{1-\gamma^-}\left[(\rho - a)(-\gamma^-)\right]^{\gamma^-}}{\left[\rho(1 - \gamma^{-1})\right]^{1-\gamma^-}}.$$