

ECON 2010c

Solution to Problem Set 6

By the Teaching Fellows for ECON 2010c

Fall 2014

1 Q-Theory with Depreciation and Taxes

- (a) The variable q represents the present discounted value of future profits from a marginal unit of installed capital. The definition here differs from that in class because the effective discount rate used to discount future marginal earnings flows is $r + \gamma$ rather than r . This accounts for the depreciation of capital at rate γ .

The value function for the problem is defined as:

$$V(k, K) = \max_I \int_t^\infty e^{-r(s-t)} \{\pi(K(s))k(s) - I(s) - C(I(s))\} ds.$$

Applying the envelope theorem, the derivative of V with respect to k is given by:

$$\frac{\partial V(k, K)}{\partial k} = \int_t^\infty e^{-r(s-t)} \pi[K(s)] \frac{\partial k(s)}{\partial k(t)} ds.$$

The differential equation $dk/du = I - \gamma k$ can be solved using an Euler integrating factor:

$$I(u)e^{\gamma(u-t)} = \left[\frac{dk}{du} + \gamma k(u) \right] e^{\gamma(u-t)} = \frac{d}{du} \{k(u)e^{\gamma(u-t)}\};$$

so that, integrating both sides gives:

$$\int_t^s I(u)e^{\gamma(u-t)} du = \int_t^s \frac{d}{du} \{k(u)e^{\gamma(u-t)}\} du = k(s)e^{\gamma(s-t)} - k(t),$$

yielding the result:

$$k(s) = k(t)e^{-\gamma(s-t)} + \int_t^s I(u)e^{-\gamma(s-u)} du.$$

It follows from $\partial k(s)/\partial k(t) = e^{-\gamma(s-t)}$ that:

$$\frac{\partial V(k, K)}{\partial k} = \int_t^\infty e^{-(r+\gamma)(s-t)} \pi[K(s)] ds = q(t).$$

(b) The Bellman equation has the form:

$$rV(k, K)dt = \max_I \{[\pi(K)k - I - C(I)]dt + dV\},$$

where dV is given by:

$$dV = \frac{\partial V}{\partial k}dk + \frac{\partial V}{\partial K}dK = q(I - \gamma k)dt + \frac{\partial V}{\partial K}dK.$$

Substituting for dV yields:

$$rV(k, K)dt = \max_I \left\{ [\pi(K)k - I - C(I)]dt + q(I - \gamma k)dt + \frac{\partial V}{\partial K}dK \right\}.$$

Thus, the first-order condition is:

$$-1 - C'(I) + q = 0 \Rightarrow q = 1 + C'(I).$$

(c) Applying the Leibniz integral rule, the time derivative of q can be expressed as:

$$\dot{q}(t) = -\pi(K(t)) + (r + \gamma) \int_t^\infty e^{-(r+\gamma)(s-t)} \pi(K(s))ds = (r + \gamma)q(t) - \pi(K(t)),$$

yielding the equation:

$$(r + \gamma)q = \pi(K) + \dot{q}.$$

The first-order condition can be written as:

$$I = C'^{-1}(q - 1).$$

Substituting for I from the equation of motion and aggregating the result over N firms yields:

$$\dot{K} = NC'^{-1}(q - 1) - \gamma K.$$

(d) From part (c), the $\dot{q} = 0$ and $\dot{K} = 0$ loci are respectively given by:

$$q = \frac{\pi(K)}{r + \gamma} \quad \text{and} \quad q = 1 + C'\left(\frac{\gamma K}{N}\right).$$

The phase diagram in Figure 1 is different from that in lecture, because the model here accounts for depreciation at rate $\gamma > 0$, leading the $\dot{K} = 0$ locus to be upward-sloping rather than horizontal.

An increase in the depreciation rate γ affects both the marginal benefit and the marginal cost of capital for a given steady-state level of aggregate capital. It reduces the NPV of future profits from a marginal unit of installed capital since one additional unit today depreciates more quickly, and it also implies higher

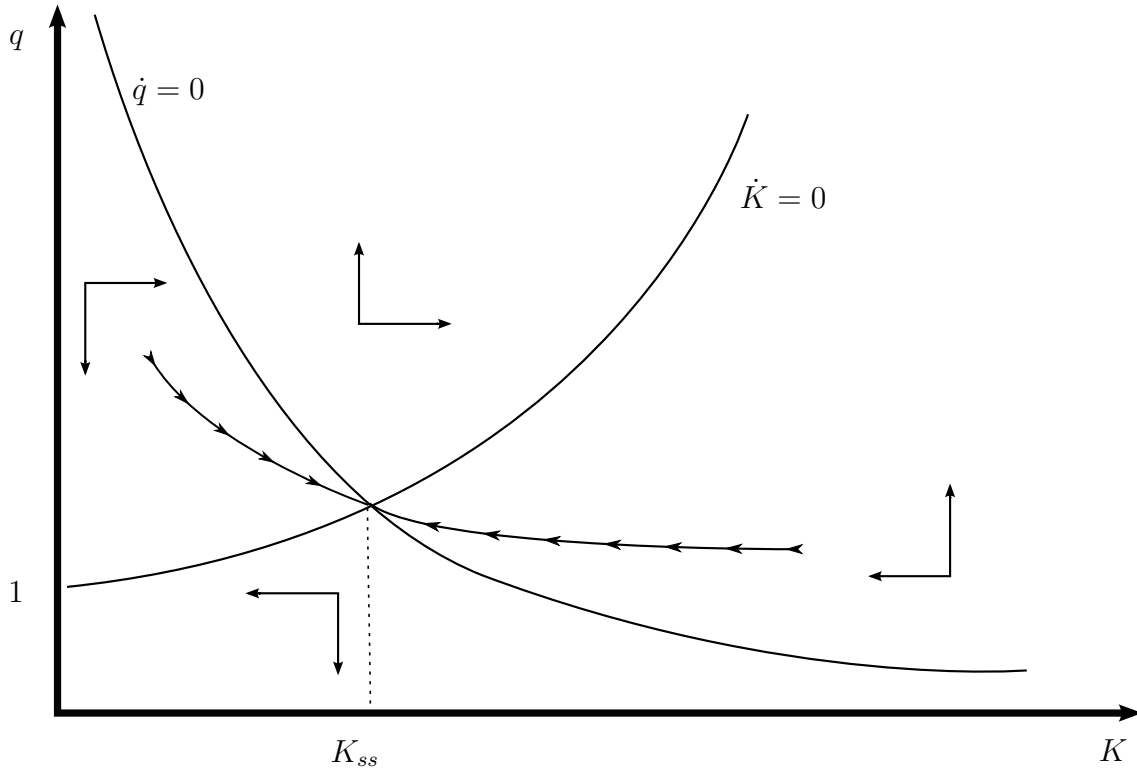


Figure 1: Phase diagram for Q-theory model with depreciation (part 1.d).

marginal cost of capital in steady state, given the convexity of the cost function and the fact that higher investment is now required in steady state to offset the depreciation of existing capital. Clearly, the steady-state level of capital will fall since, for given K , the marginal benefit of capital is now lower and its marginal cost higher. These effects correspond to inward shifts of both loci in the phase diagram of Figure 1. We can also derive this analytically by calculating the comparative static:

$$\begin{aligned} \frac{\pi(K)}{r+\gamma} - 1 - C'\left(\frac{\gamma K}{N}\right) &= 0 \\ \Rightarrow \frac{\partial K}{\partial \gamma} &= -\frac{-\frac{\pi(K)}{(r+\gamma)^2} - \frac{K}{N}C''\left(\frac{\gamma K}{N}\right)}{\frac{\pi'(K)}{r+\gamma} - \frac{\gamma}{N}C''\left(\frac{\gamma K}{N}\right)} < 0 \end{aligned}$$

(e) The Bellman equation between times t_0 and t_1 is given by:

$$rV(k, K)dt = \max_I \left\{ [\pi(K)k - (1-\theta)I - C(I)]dt + q(I - \gamma k)dt + \frac{\partial V}{\partial K}dK \right\};$$

so that, the first-order condition yields:

$$-(1-\theta) - C'(I) + q = 0 \Rightarrow q = 1 - \theta + C'(I).$$

The equations governing the dynamics of K and q are the same as before, except that the dynamics of K between times t_0 and t_1 are now given by:

$$\dot{K} = NC'^{-1}(q - 1 + \theta) - \gamma K.$$

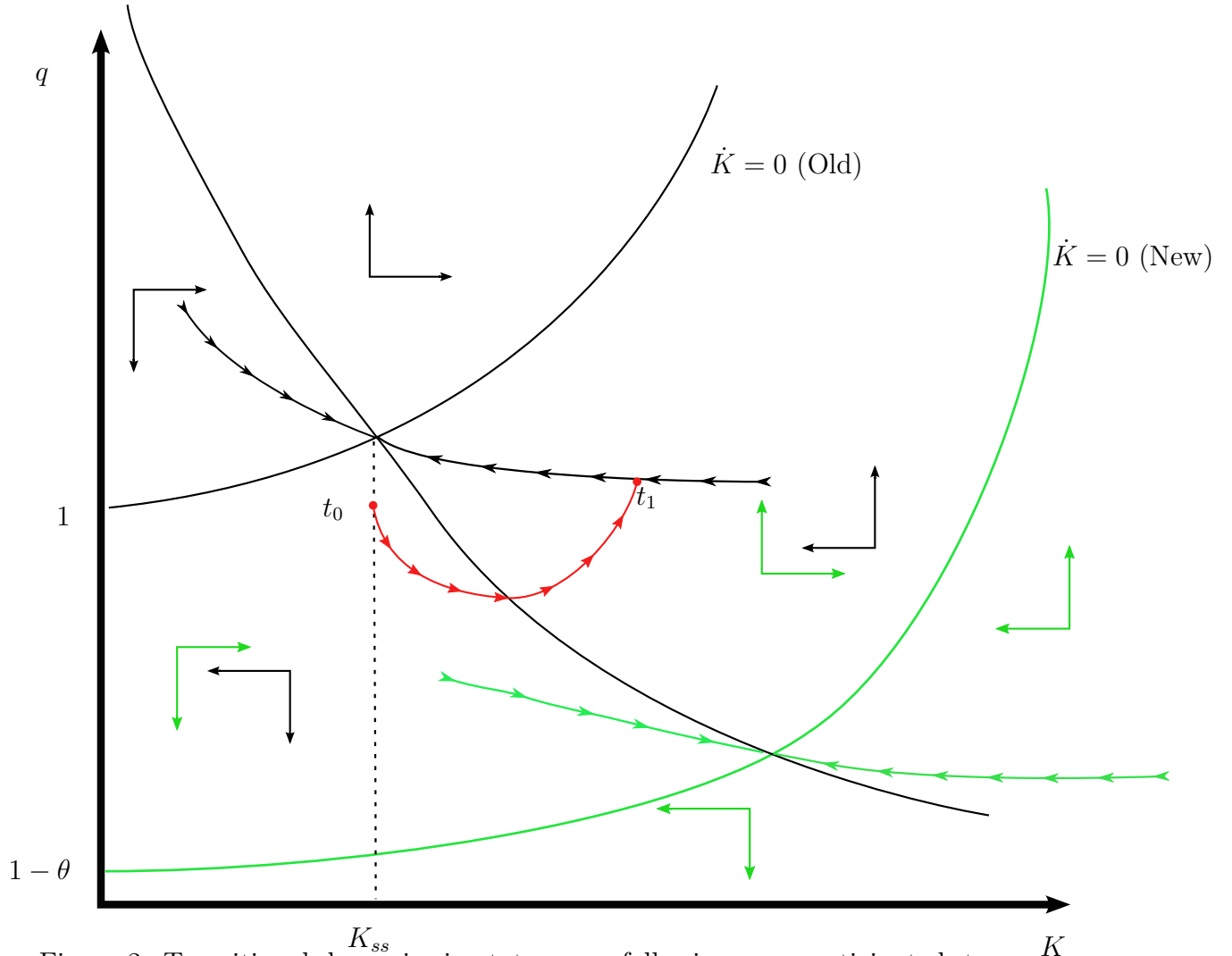


Figure 2: Transitional dynamics in state space following an unanticipated, temporary investment tax credit policy (part 1.f.)

- (f) The transitional dynamics following a temporary policy change, the end date of which is fixed and commonly known, is in general determined from the following two equilibrium considerations: first, prices cannot be *expected* to jump (as this would imply an arbitrage opportunity); second, at the time of expiration of the temporary policy the economy must be on the “old” saddle path (stable manifold) of the economy. The latter holds by the standard argument that any path

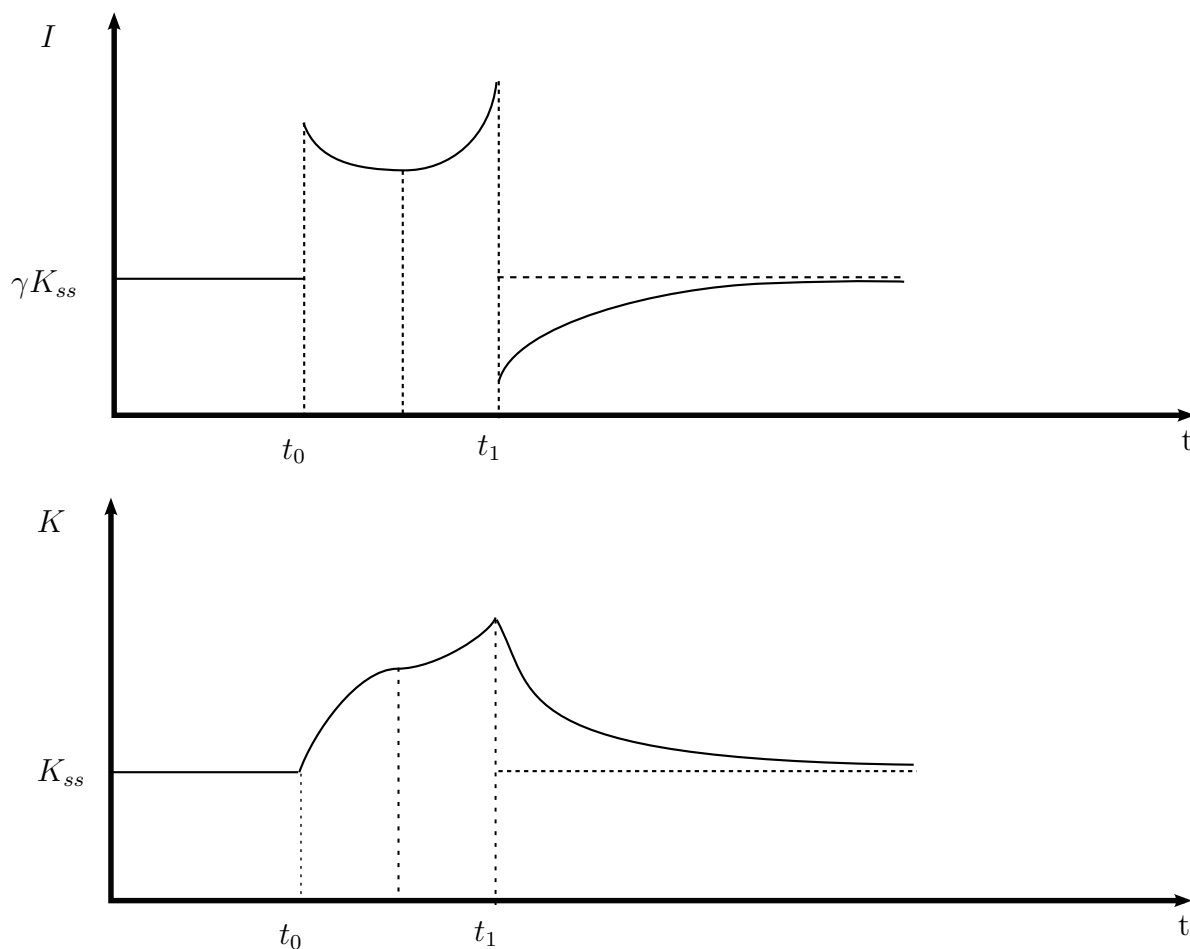


Figure 3: Time dynamics of investment and capital following an unanticipated, temporary investment tax credit policy (part 1.f.)

not on the saddle path is inconsistent with equilibrium, as it violates either the transversality condition or optimality of investment choices.

Consider Figure 2. The two considerations above imply that the economy must follow a path that is continuous (for $t > t_0$) and leads it to a point on the old saddle path at time t_1 while following the temporary dynamics associated with the investment tax credit policy. First, note that at time t_0 the economy cannot jump to a point above the old steady state¹ because it would be impossible for the economy to return to the old saddle path in the future, as can be seen from the direction arrows at this region (which coincide for the original and temporary laws of motion). For the same reason, the economy cannot jump to a point below the “new” saddle path (the saddle path of an economy where the investment tax credit is permanent). Thus, the economy will jump down to a point above the

¹The jump at time t_0 must be vertical, since capital is a stock variable and thus cannot jump.

new saddle path, eventually crossing the $\dot{q} = 0$ locus and entering the region where both capital and price are increasing, bringing the economy back to the old saddle path. Note that capital increases between t_0 and t_1 and decreases from t_1 onwards while the economy converges asymptotically to the old steady state. The price of capital follows a nonmonotonic path while the temporary regime is in effect, first declining until a point in time strictly between t_0 and t_1 and increasing thereafter. Noting from the firms' first-order condition that investment follows the same dynamics qualitatively as the price of capital,

$$\begin{aligned} q(t) &= 1 - \theta + C'(I(t)) \quad \forall t \\ \Rightarrow \dot{q}(t) &= \underbrace{C''(I(t))}_{(+)} \dot{I}(t) \end{aligned}$$

we see that investment is nonmonotonic over the interval $[t_0, t_1]$.

What is the intuition behind these dynamics? First, note that, since the policy reduces the marginal cost of investment, the latter will be higher (compared to its old steady-state level) and thus aggregate capital will be increasing while the policy is in effect. The (rational) expectation of higher future aggregate capital is reflected in the price of capital at t_0 : because the gain from an additional unit of capital depends negatively on the level of aggregate capital ($\pi'(K) < 0$), q , which is the NPV of the stream of current and future profits, must decline and its level will always be lower than its steady-state level.²

The reason behind the nonmonotonic dynamics of $q(t)$ and thus investment is more subtle. There are two opposing forces affecting the time evolution of the NPV of future marginal gains after time t_0 . On one hand, aggregate capital is increasing, which implies declining current and (short-term) future marginal profits. On the other hand, the time when capital starts declining gets closer as time passes, which implies increasing future marginal profits. The first force dominates initially, but the second one takes over when the economy crosses the $\dot{q} = 0$ locus.

At time t_1 the economy reverts to its old dynamics, so that investment must now be below its steady-state value despite the continued rise of q . Figure 3 depicts the time dynamics of capital and investment. Note that, without specific assumptions on the functional forms for $\pi(\cdot)$ and $C(\cdot)$, we can only pin down the level of investment relative to its steady-state level and the sign of the first derivative of investment (whether it is sloping upwards or downwards), but not whether investment is convex or concave from t_0 to t_1 (or whether its left limit at t_1 is higher than its right limit at t_0). We can, however, pin down the concavity of capital: capital has kinks at t_0 and t_1 (since its first derivative, net investment, jumps), and it is concave from t_0 until the time the economy crosses the $\dot{q} = 0$ locus and convex thereafter. Finally, since we know that the economy will reach

²Note that the economy reaches the steady state again only asymptotically.

its old steady state only asymptotically, we know that investment will be concave and capital convex after t_1 .

If the policy change was permanent the economy would jump down to the new saddle path at t_0 , which is below the point depicted in Figure 2 for the temporary policy case. Since $q(t_0)$ is now lower, investment will rise by less than in the temporary policy case (for given θ). Intuitively, the convexity of adjustment costs implies that agents want to smooth their investment over time. But when the policy change is temporary they must make their investment during a short (and finite) period of time in order to take advantage of the investment credit. Therefore, they will optimally choose a higher short-term investment rate following the announcement.

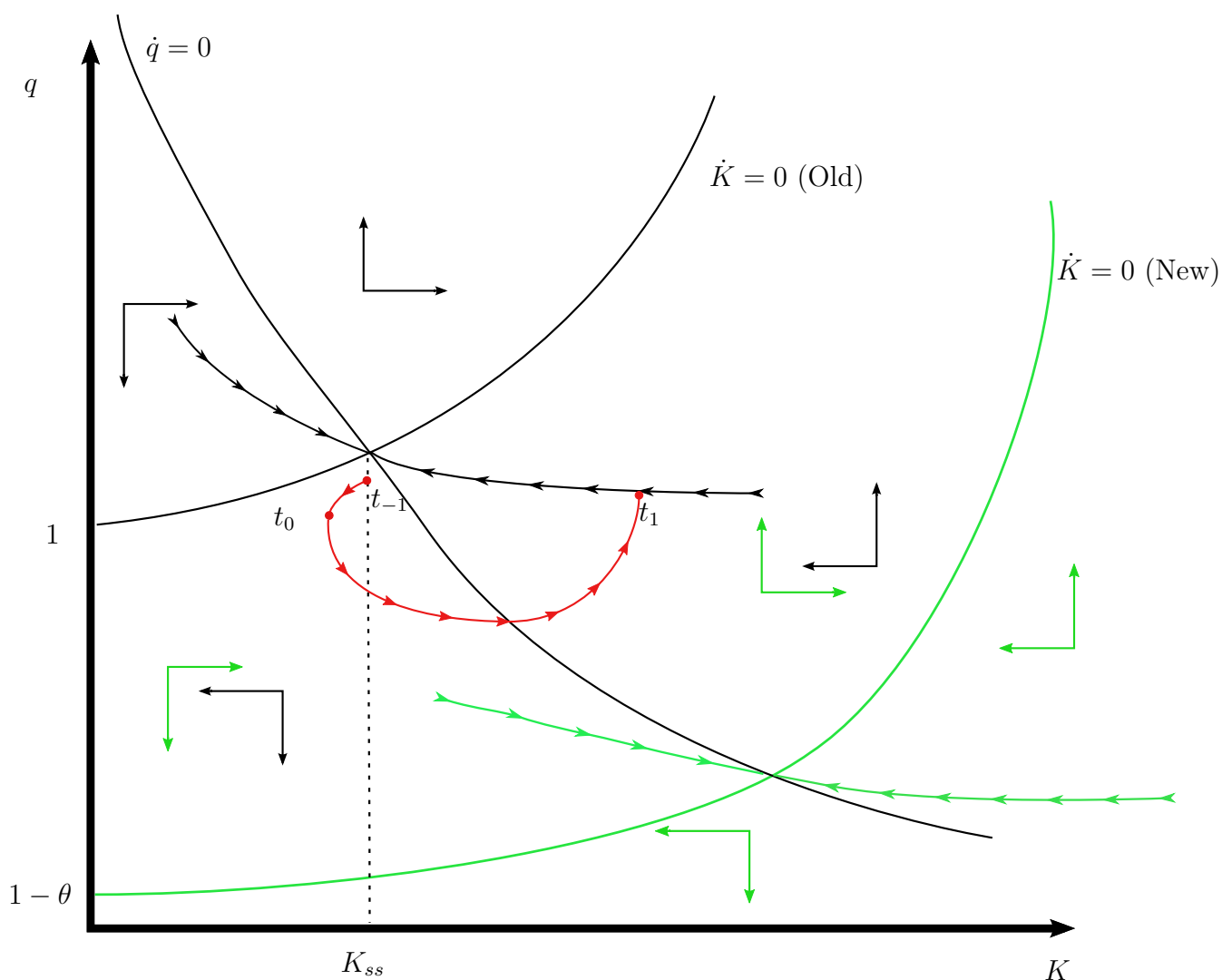


Figure 4: Transitional dynamics in state space following an anticipated, temporary investment tax credit policy (part 1.g.)

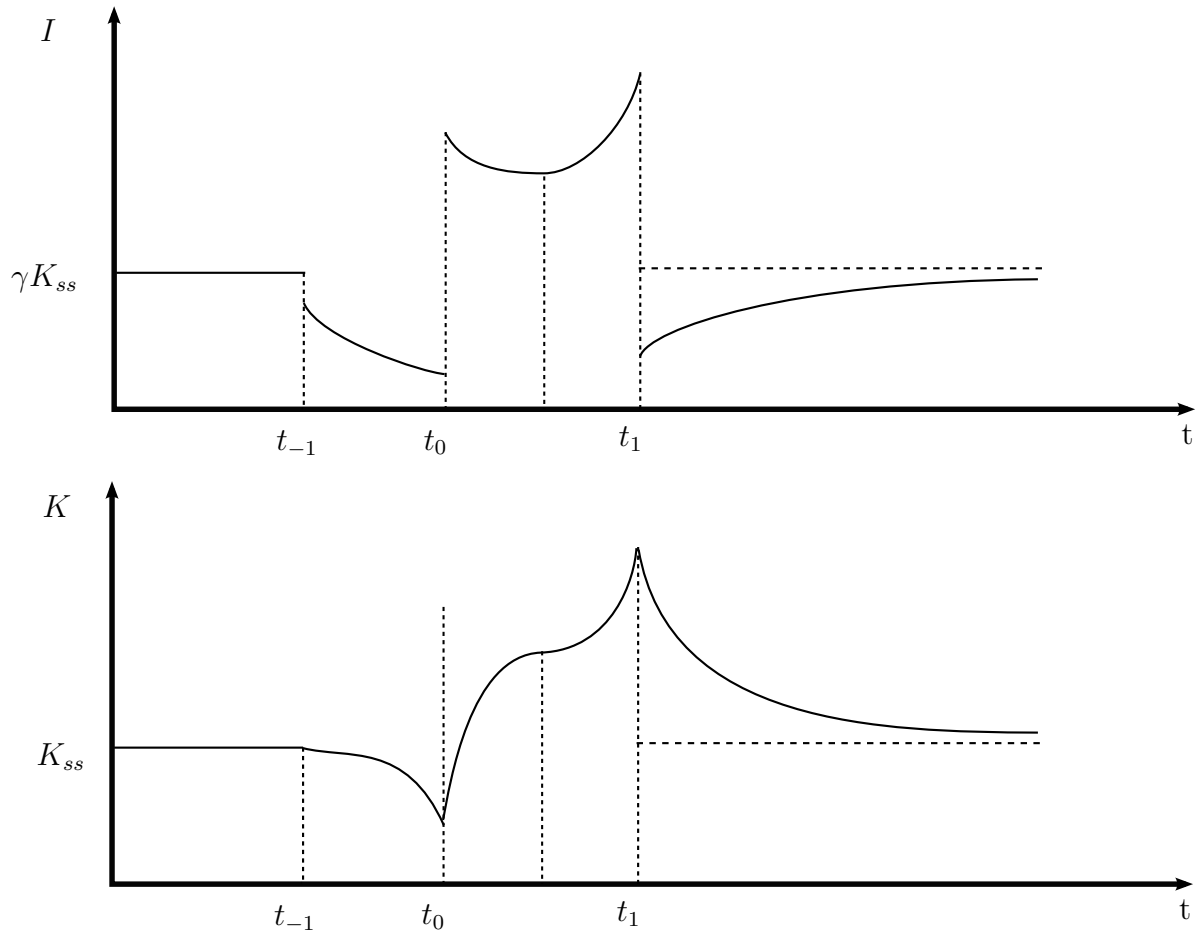


Figure 5: Time dynamics of investment and capital following an anticipated, temporary investment tax credit policy (part 1.g.)

- (g) When the temporary policy change is anticipated, the two equilibrium considerations mentioned in the previous part still hold: the price of capital cannot be expected to jump, so it will jump only at time t_{-1} ; and the economy must lie at a point on the old saddle path at time t_1 . The dynamics are depicted in figure 4. The difference is that now the economy will obey the old dynamics from t_{-1} to t_0 (depicted by the black direction arrows) and the new dynamics from t_0 to t_1 (depicted by the green direction arrows). Therefore, capital and its price will both be decreasing following the announcement. At time t_0 capital will start to increase while q will keep decreasing. The dynamics after t_0 are qualitatively the same as in the case of an unanticipated policy change.

Intuitively, agents at time 0 expect that they will be able to invest more cheaply in the future and are therefore willing to postpone part of the investment they would undertake today absent the announcement. Because a level of investment below $\gamma K(t)$ implies that investment net of depreciation is negative, capital will

be declining initially.

The time dynamics of investment and capital are depicted in Figure 5. Note that, even though the capital price will jump only once (always at the time of the announcement), investment will jump three times, at t_{-1} , t_0 and t_1 . The fact that investment is declining from t_{-1} to t_0 implies that capital declines in a concave manner over this time interval.

2 True, False, or Uncertain

- (a) False. Letting $z(t)$ be a Wiener process, suppose that $x(t)$ follows the Ito process $dx = a(x, t)dt + b(x, t)dz$, and consider a function $V(x, t)$ that is twice differentiable in x and once in t . Then Ito's lemma indicates that $V(x, t)$ follows the stochastic process:

$$dV = \left[\frac{\partial V}{\partial t} + a(x, t) \frac{\partial V}{\partial x} + \frac{1}{2} b^2(x, t) \frac{\partial^2 V}{\partial x^2} \right] dt + b(x, t) dz.$$

This formula reflects the fact that the term $(dz)^2$ can be treated as a term of order dt when calculating the total derivative of V .

- (b) False. See question 1.g. for the correct dynamics and intuition.

3 Bertola and Caballero (1990)

- (a) For the investment case, X could represent the deviation of a firm's capital stock from its optimal level. The motivation for the cost structure here is that while there are marginal costs of acquiring or selling capital, there are also often fixed transaction/installation costs for new capital investments (e.g., machinery, buildings, etc.). In addition to costs of physically installing the capital or maybe broker's fees for investing in real estate, one could also consider fixed costs of searching, for instance, for the best type of machine to purchase. Likewise, these fixed transaction/de-installation costs may exist for negative investment as well when capital goods must be uninstalled and sold.

Another possible application of this model is where X might represent the deviation of a firm's labor force from the optimal level. Because firms often need to screen and train new workers, it can be costly for a firm to make upward adjustments to its labor force. Employment contracts might also contain provisions that penalize firms for terminating employees; thus, downward adjustments to the labor force can also be costly. If even small changes in the labor force are costly, then the firm might be unwilling to hire or fire workers if employment deviates only slightly from its optimal level.

- (b) $u \leq d$ because the total adjustment cost is weakly increasing in the size of the adjustment. Suppose, for the sake of contradiction, that the agent optimally sets $u > d$. Then, it must be more profitable to be at u than to be at d ; otherwise, the firm would choose to adjust to d since doing so would be weakly less costly given that $d < u \Rightarrow d - U < u - U$. But then $d < u \Rightarrow D - u < D - d$, so the firm would prefer to adjust from D to u than from D to d since adjusting from D to u would be less costly, a contradiction.

If there are no variable costs of adjustment, then the size of an adjustment does not affect the cost of the adjustment; thus, conditional on making an adjustment, the firm would always seek to adjust to the same optimal point. This follows from the fact that, given the assumed flow payoff function, no two points in state space are equally profitable. Thus, $u = d$ if $c_u = c_d = 0$.

The points u and d must both lie in the interval $[U, D]$, because it is unprofitable for a firm to adjust X to a level outside the acceptable interval $[U, D]$. In particular, the firm would otherwise need to make an additional costly adjustment after an initial adjustment, whereas the ultimate point of the adjustment process could be reached at lower cost with a single adjustment. It follows that $U \leq u \leq d \leq D$.

If there are no fixed costs of adjustment, then the firm makes an infinitesimal adjustment when the marginal cost of an adjustment falls just below the expected loss from a marginal change in X . Thus, $u = U$ and $d = D$ if $C_U = C_D = 0$.

- (c) The discrete-time approximation of the Bellman equation is as follows:

$$\begin{aligned} V(X) &= -\frac{b}{2}X^2\Delta t + (1 + \rho\Delta t)^{-1}E[V(X + \Delta X)] \\ \Rightarrow (1 + \rho\Delta t)V(X) &= -\frac{b}{2}X^2(1 + \rho\Delta t)\Delta t + E[V(X + \Delta X)] \\ \Rightarrow \rho V(X)\Delta t &= -\frac{b}{2}X^2(1 + \rho\Delta t)\Delta t + E[\Delta V(X)]. \end{aligned}$$

Dividing both sides of the equation by Δt and taking the limit as Δt goes to zero yields:

$$\rho V(X)dt = -\frac{b}{2}X^2dt + E[dV(X)].$$

By Ito's lemma,

$$E(dV) = (\alpha V' + \frac{1}{2}\sigma^2 V'')dt,$$

Thus, the Bellman equation can be written as:

$$\rho V = -\frac{b}{2}X^2 + \alpha V' + \frac{1}{2}\sigma^2 V''.$$

The term $\frac{\partial V}{\partial t}$ does not appear in the Bellman equation because the problem is stationary, meaning that the value function does not depend directly on time, given the current state X . The Bellman equation sets the required return ρV on an asset equal to the sum of the instantaneous dividend $-bX^2/2$ and the expected capital gain $E(dV)$, which can be further decomposed into a drift term $\alpha V'$ and a volatility term $\sigma^2 V''/2$.

(d) The function $V(X)$ given by:

$$V(X) = -\frac{b}{2} \left(\frac{X^2}{\rho} + \frac{\sigma^2 + 2\alpha X}{\rho^2} + \frac{2\alpha^2}{\rho^3} \right)$$

has the following first and second derivatives:

$$V'(X) = -b \left(\frac{X}{\rho} + \frac{\alpha}{\rho^2} \right), \quad V''(X) = -\frac{b}{\rho}.$$

Substituting into the Bellman equation yields:

$$-\rho \frac{b}{2} \left(\frac{X^2}{\rho} + \frac{\sigma^2 + 2\alpha X}{\rho^2} + \frac{2\alpha^2}{\rho^3} \right) = -\frac{b}{2} X^2 - \alpha b \left(\frac{X}{\rho} + \frac{\alpha}{\rho^2} \right) - \frac{1}{2} \sigma^2 \frac{b}{\rho},$$

which holds for all values of X , confirming that the function V defined above is a solution to the Bellman equation.

If the adjustment cost is infinite the firm never makes an adjustment; hence, the expected present value of the firm's payoffs is given by:

$$V[X(0)] = E \left[-\frac{b}{2} \int_0^\infty e^{-\rho t} [X(t)]^2 dt \right] = E \left[-\frac{b}{2} \int_0^\infty e^{-\rho t} [X(0) + \alpha t + \sigma z(t)]^2 dt \right].$$

Because $z(t) \sim N(0, t)$ is a Wiener process, $V[X(0)]$ can be calculated as follows:

$$\begin{aligned} V[X(0)] &= -\frac{b}{2} \int_0^\infty e^{-\rho t} \{ [X(0) + \alpha t]^2 + \sigma^2 t \} dt \\ &= -\frac{b}{2} \int_0^\infty e^{-\rho t} \{ [X(0)]^2 + [\sigma^2 + 2\alpha X(0)]t + \alpha^2 t^2 \} dt \\ &= -\frac{b}{2} \left\{ [X(0)]^2 \int_0^\infty e^{-\rho t} dt + [\sigma^2 + 2\alpha X(0)] \int_0^\infty t e^{-\rho t} dt + \alpha^2 \int_0^\infty t^2 e^{-\rho t} dt \right\} \\ &= -\frac{b}{2} \left\{ [X(0)]^2 \int_0^\infty e^{-\rho t} dt - [\sigma^2 + 2\alpha X(0)] \frac{d}{d\rho} \int_0^\infty e^{-\rho t} dt + \alpha^2 \frac{d^2}{d\rho^2} \int_0^\infty e^{-\rho t} dt \right\} \\ &= -\frac{b}{2} \left\{ [X(0)]^2 \left(\frac{1}{\rho} \right) - [\sigma^2 + 2\alpha X(0)] \frac{d}{d\rho} \left(\frac{1}{\rho} \right) + \alpha^2 \frac{d^2}{d\rho^2} \left(\frac{1}{\rho} \right) \right\} \\ &= -\frac{b}{2} \left(\frac{[X(0)]^2}{\rho} + \frac{\sigma^2 + 2\alpha X(0)}{\rho^2} + \frac{2\alpha^2}{\rho^3} \right). \end{aligned}$$

(e) The Bellman equation provides a second-order linear differential equation for V . Because the homogeneous equation is:

$$\frac{1}{2} \sigma^2 V''(X) + \alpha V'(X) - \rho V(X) = 0,$$

we seek a solution of the form $V(X) = Ae^{rX}$. Substituting into the homogeneous equation yields:

$$\frac{1}{2}\sigma^2 r^2 Ae^{rX} + \alpha r Ae^{rX} - \rho Ae^{rX} = 0 \Rightarrow \sigma^2 r^2 + 2\alpha r - 2\rho = 0 \Rightarrow r = \frac{-\alpha \pm \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}.$$

Denoting the roots of the quadratic equation by $\alpha_1 > 0$ and $r_2 < 0$ as in the problem statement, the general solution of the differential equation is:

$$V(X) = -\frac{b}{2} \left(\frac{X^2}{\rho} + \frac{\sigma^2 + 2\alpha X}{\rho^2} + \frac{2\alpha^2}{\rho^3} \right) + A_1 e^{\alpha_1 X} + A_2 e^{\alpha_2 X}.$$

(f) For $X < U$ or $X > D$, the value function V must satisfy:

$$V(X) = \begin{cases} V(u) - [C_U + c_U(u - X)], & X < U \\ V(d) - [C_D + c_D(X - d)], & X > D \end{cases}, \quad (1)$$

because the value of being at state $X < U$ or $X > D$ must equal the difference between the value of the problem once the desired adjustment has been made and the cost of making such an adjustment. Differentiating with respect to X results in:

$$V'(X) = \begin{cases} c_U, & X < U \\ -c_D, & X > D \end{cases}. \quad (2)$$

Because a firm at U or D must be indifferent between remaining in the current state and adjusting to u or d , the value-matching conditions are:

$$V(U) = V(u) - [C_U + c_U(u - U)] \text{ and } V(D) = V(d) - [C_D + c_D(D - d)]. \quad (3)$$

Since U and D are chosen optimally, we have the smooth-pasting conditions:

$$V'(U) = c_U \text{ and } V'(D) = -c_D. \quad (4)$$

In addition, the firm must choose the points u and d optimally, conditional on making an adjustment. Thus, u and d must solve:

$$u = \arg \max \{V(u) - [C_U + c_U(u - X)]\} \text{ and } d = \arg \max \{V(d) - [C_D + c_D(X - d)]\},$$

yielding the first-order conditions:

$$V'(u) = c_U \text{ and } V'(d) = -c_D. \quad (5)$$

In summary, (1) and (2) complete our characterization of V in the action regions while (3), (4), and (5) give us the six conditions needed to solve for the six unknowns: U , u , d , D , A_1 , and A_2 .

- (g) A increase in σ raises the option value of postponing an adjustment at a given value of X ; so that, U decreases and D increases. In particular, a higher value of σ makes it more likely that a random change moves the process closer to its target value, thereby increasing the firm's willingness to refrain from making an adjustment to the process. Even though it also becomes more likely that a random change moves the process farther from its target, the firm always has the option of adjust in the latter case.
- (h) If it were the case that $U > 0$ or $D < 0$, then the optimal value of X would lie outside the region of inaction. That is, as we approached the optimal value of $X = 0$, we would always end up adjusting away from it. Because this is contradictory, we must have $U \leq 0 \leq D$. If the drift term α is negative and large, then it is possible that $u > 0$, so as to compensate for the expected decline in X without the need to revise X upward again soon afterwards. For the same reason, it is possible that $d < 0$ if α is positive and large. Of course, this requires that fixed costs are nonzero; if $C_U = 0$ then $u = U \leq 0$ and similarly for d .