

Problem Set #1
Due: 11 September 2014

Instructor: David Laibson
Economics 2010c

Problem 1 (Growth Model): Recall the growth model that we discussed in class. We expressed the sequence problem as

$$v(k_0) = \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(k_t^\alpha - k_{t+1})$$

subject to the constraint

$$k_{t+1} \in [0, k_t^\alpha] \equiv \Gamma(k_t).$$

Consider the associated Bellman equation

$$v(k) = \sup_{y \in \Gamma(k)} \ln(k^\alpha - y) + \beta v(y).$$

Finally, note that $0 \leq \alpha < 1$.

- a. [You'll need material from Lecture 2 for part (a). If you want to start the problem set before Lecture 2, just jump directly to part (b) and continue from there.] Consider the Bellman (functional) operator, T , defined by

$$(Tf)(k) = \sup_{y \in \Gamma(k)} \ln(k^\alpha - y) + \beta f(y).$$

Let $\hat{v}(k) = \frac{\alpha \ln(k)}{1 - \alpha\beta}$. Show that

$$(T^n \hat{v})(k) = \frac{1 - \beta^n}{1 - \beta} \left[\ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \ln(\alpha\beta) \right] + \frac{\alpha \ln(k)}{1 - \alpha\beta}.$$

To prove this you'll need to show that $y = \alpha\beta k^\alpha$, and substitute this expression into the functional operator. Let,

$$\lim_{n \rightarrow \infty} (T^n \hat{v})(k) = v(k).$$

Confirm that $v(k)$ is a solution to the the functional equation. You have now solved the functional equation by iterating the operator T on a starting guess.

- b.** Try to solve the Bellman Equation by “guessing” a solution. Specifically, start by guessing that the form of the solution is

$$v(k) = \psi + \phi \ln(k).$$

We will solve for the coefficients ψ and ϕ , and show that $v(k)$ solves the functional equation. Rewrite the functional equation substituting in $v(k) = \psi + \phi \ln(k)$. Use the Envelope Theorem (ET) and the First Order Condition (FOC) to show

$$\phi = \frac{\alpha}{1 - \alpha\beta}.$$

Now use the FOC to show

$$y = \alpha\beta k^\alpha.$$

Finally, show that the functional equation is satisfied at all feasible values of k_0 if

$$\psi = \frac{1}{1 - \beta} \left[\ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \ln(\alpha\beta) \right].$$

You have now solved the functional equation by using the guess and check method.

- c.** We have derived the policy function:

$$y = g(k) = \alpha\beta k^\alpha.$$

Derive the optimal sequence of state variables $\{k_t^*\}_0^\infty$ which would be generated by this policy function. Show that

$$v(k_0) = \sum_{t=0}^{\infty} \beta^t \ln([(k_t^*)^\alpha - k_{t+1}^*]),$$

thereby confirming that this policy function is optimal.

- d.** (Optional problem for students who want to be challenged and have an interest in growth theory.) Show that the steady state capital stock is given by:

$$\alpha\beta k^{\alpha-1} = 1.$$

Now, linearize the equilibrium policy function in a neighborhood of the steady state. You should find:

$$\frac{k_{t+1} - k_{steadystate}}{k_t - k_{steadystate}} = e^{-(-\ln \alpha)},$$

implying that the convergence rate is $-\ln(\alpha)$. Explain why α is the capital share in this economy. Most economists think that the capital share lies somewhere between 0.3 (the capital share for physical capital), and 0.7 (the capital share for physical and human capital). What do these capital shares imply for the convergence rate? In the data, the measured convergence rate tends to be below 0.05. Why aren't we matching the data? (Hint: think about the depreciation rate which has been implicitly assumed in the model above. What depreciation rate did we implicitly assume and why does it speed up the rate of convergence?)

Problem 2 (Equity model): Assume that a consumer with only equity wealth must choose period by period consumption in a discrete-time dynamic optimization problem. Specifically, consider the sequence problem:

$$v(x_0) = \sup_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \exp(-\rho t) u(c_t)$$

subject to the constraints: $x_{t+1} = \exp(r + \sigma u_t - \sigma^2/2)(x_t - c_t)$, u_t iid and $u_t \sim N(0, 1)$, $c_t \in [0, x_t]$, $x_0 > 0$. Here x_t represents equity wealth at period t and c_t represents consumption at period t . The consumer has discount rate ρ and the consumer can only invest in a risky asset with expected return $\exp(r) = E \exp(r + \sigma u - \sigma^2/2)$. Finally, assume that the consumer has an isoelastic utility function:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

with $\gamma \in [0, \infty]$. Note that this utility function has the property of constant relative risk aversion

$$\frac{-cu''(c)}{u'(c)} = \gamma.$$

This homotheticity property enables us to analytically solve this problem.

The associated Bellman equation for this problem is given by

$$v(x) = \sup_{y \in [0, x]} u(x - y) + E \exp(-\rho) v(\exp(r + \sigma u - \sigma^2/2)y).$$

- a. Explain all of the terms in the Bellman equation. Make sure that this equation makes sense to you.
- b. Now guess that the value function takes the special form

$$v(x) = \phi \frac{x^{1-\gamma}}{1-\gamma}.$$

Note the close similarity between this functional form and the functional form of the utility function. Assuming that the value function guess is correct, use the Envelope Theorem to derive the consumption function:

$$c = \phi^{-\frac{1}{\gamma}} x.$$

Now verify that the Bellman Equation is satisfied for a particular value of ϕ . Do **not** solve for ϕ (it's a very nasty expression). Instead, show that

$$\ln(1 - \phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma} [(1 - \gamma)r - \rho] + \frac{1}{2}(\gamma - 1)\sigma^2$$

- c. Now consider the natural log of the ratio of c_{t+1} and c_t . Show that

$$E \ln \left(\frac{c_{t+1}}{c_t} \right) = \frac{1}{\gamma} (r - \rho) + \frac{\gamma}{2} \sigma^2 - \sigma^2.$$

- d. Interpret the previous equation for the certainty case $\sigma = 0$. Note that $\ln \left(\frac{c_{t+1}}{c_t} \right) = \Delta \ln c_{t+1}$ is the growth rate of consumption. Explain why $\Delta \ln c_{t+1}$ increases in r and decreases in ρ . Why does the coefficient of relative risk aversion, γ , appear in the denominator of the expression? Why does the coefficient of relative risk aversion regulate the consumer's willingness to substitute consumption between periods?
- e. (Very interesting optional question for students who want to be challenged and are interested in finance.) Suppose a bond with a sure payoff were added to this economy. Assume the bond pays off ε dollars in perpetuity, where ε is small. What will the equilibrium interest rate be

on this bond? For starters, how will the bond interest rate compare to the interest rate on stocks? Can you derive a closed form expression for the bond interest rate? What is the marginal utility of a marginal sure payoff next period? How much marginal consumption would you give up today to get such a sure marginal payoff tomorrow. We'll come back to this question later in the course. But, for those of you who want a challenge, think about the bond problem now.

Problem 3 (True/False/Uncertain): I often give T/F/U questions on exams. Such questions are short and enable me to cover a lot of different topics quickly. That way the coverage of the exam is diversified, rather than being concentrated on one or two long questions. T/F/U questions focus on the key ideas and force you to demonstrate a conceptual understanding. These questions are graded on the quality of your explanation (not on the one-word answer itself). So explain each answer. You may or may not want to use formal/rigorous mathematical proof to support your answer. However, when a statement is false, you'll get full credit only if you provide a counter-example.

- a. All supremum sequence problems have a unique value-function solution. (Hint: this is true. Why?)
- b. If the flow payoff function is bounded, then there exists a unique bounded solution to the Bellman Equation.
- c. Let $v(\cdot)$ be a solution to a Bellman Equation for some dynamic optimization problem. If the flow payoff function, $F(\cdot, \cdot)$, is unbounded, then it is always possible to find a feasible sequence of actions such that

$$\lim_{n \rightarrow \infty} \beta^n v(x_n) \neq 0.$$

Hint: this is *false*. Can you generate a counterexample? In other words, can you generate an example for which the flow payoff function is unbounded, but it is *impossible* to generate a feasible sequence of actions such that $\lim_{n \rightarrow \infty} \beta^n v(x_n) \neq 0$.

- d. In the growth problem (problem 1), for any $\varepsilon > 0$, there exists a value of T , such that $k_t < 1 + \varepsilon$ for all $t > T$. Hint: this is true.
- e. In the growth problem (problem 1), $\lim_{n \rightarrow \infty} \beta^n v(x_n) \leq 0$.