Problem Set #2 Instructor: David Laibson
Due in class on Thursday (9/18) Economics 2010c

Problem 1 (**Eat-the-Pie Problem**): Consider the following sequence problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(c_t)$$

subject to the constraints:

$$W_{t+1} = R(W_t - c_t)$$

$$0 \leq c_t \leq W_t$$

$$W_0 > 0 \quad \text{given.}$$

I'm now going to ask you to analyze this problem. This analysis provides a quick review of concepts that should by now be familiar. I will put problems like this on the final exam.

a. Motivate the economic problem above. Evaluate the implicit assumptions. What is economically sensible and what is not sensible about this modeling set-up?

b. Explain why the Bellman equation for this problem is given by:

$$v(W) = \sup_{c \in [0,W]} \{u(c) + \delta v(R(W-c))\} \quad \forall W$$

Why doesn't an expectation operator need to appear in this Bellman equation?

c. Using Blackwell's sufficiency conditions, prove that the Bellman operator, B,

$$(Bf)(W) = \sup_{c \in [0,W]} \{u(c) + \delta f(R(W-c))\} \quad \forall W$$

is a contraction mapping. You should assume that u is a bounded function. (Why is this boundedness assumption necessary for the application of Blackwell's Theorem?) Explain what the contraction mapping property implies about iterative solution methods.

d. Now assume that,

$$u(c) = \left\{ \begin{array}{ll} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \ln c & \text{if } \gamma = 1 \end{array} \right\}.$$

(So u is no longer bounded.) Use the guess method to solve the Bellman equation. Specifically, guess the form of the solution.

$$v(W) = \left\{ \begin{array}{ll} \psi \frac{W^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \phi + \psi \ln W & \text{if} \quad \gamma = 1 \end{array} \right\}$$

Confirm that this solution works.

e. Derive the optimal policy rule:

$$c = \psi^{-\frac{1}{\gamma}} W$$

$$\psi^{-\frac{1}{\gamma}} = 1 - (\delta R^{1-\gamma})^{\frac{1}{\gamma}}$$

Note that this rule applies for all values of γ.

f. When $\gamma = 1$ the consumption rule collapses to $c = (1-\delta)W$. Why does consumption no longer depend on the value of the interest rate? (Hint: think about income effects and substitution effects.)

Problem 2 (**Growth Model**): Recall the growth model that we discussed in class. We expressed the sequence problem as

$$v(k_0) = \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(k_t^\alpha - k_{t+1})$$

subject to the constraint

$$k_{t+1} \in [0, k_t^\alpha] \equiv \Gamma(k_t).$$

Consider the associated Bellman equation

$$v(k) = \sup_{y \in \Gamma(k)} \ln(k^\alpha - y) + \beta v(y).$$

Finally, note that $0 \leq \alpha < 1$. Consider the Bellman (functional) operator, T, defined by

$$(Tf)(k) = \sup_{y \in \Gamma(k)} \ln(k^\alpha - y) + \beta f(y).$$

Let $\hat{v}(k) = \frac{\alpha \ln(k)}{1-\alpha\beta}$. Show that

$$(T^n \hat{v})(k) = \frac{1-\beta^n}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right] + \frac{\alpha \ln(k)}{1-\alpha\beta}.$$

To prove this you'll need to show that $y = \alpha\beta k^\alpha$, and substitute this expression into the functional operator. Let,

$$\lim_{n\to\infty}(T^n \hat{v})(k) = v(k).$$

Confirm that $v(k)$ is a fixed point of the functional equation. You have now solved the Bellman equation by iterating the operator T on a starting guess.

Problem 3 (**Search and Optimal Stopping**): Reconsider the optimal stopping problem from lecture 2:

Example *Each period (over an infinite horizon) the consumer draws a job offer from a uniform distribution with support in the unit interval: $x \sim u[0,1]$. The consumer can either accept the offer and realize NPV x, or the consumer can wait another period and draw again. Once you accept an offer the game ends. Waiting to accept an offer is costly because the* *value of the remaining offers declines at rate $\rho = -\ln\delta$ between periods. The Bellman equation for this problem is:*

$$v(x) = \max\{x, \delta E v(x_{+1})\}$$

a. Explain the intuition behind the Bellman Equation.

b. Consider the associated functional operator:

$$(Bw)(x) = \max\{x, \delta E w(x_{+1})\} \quad \forall x.$$

Using Blackwell's conditions, show that this Bellman operator is a contraction mapping.

c. What does the contraction property imply about $\lim_{n\to\infty} B^n w$, where w is an arbitrary function?

d. Let $w(x) = 1 \ \forall x$. *Analytically*, iterate $B^n w$, and show that

$$\lim_{n\to\infty}(B^n w)(x) = v(x) = \left\{ \begin{array}{ll} x^* & \text{if} \quad x \le x^* \\ x & \text{if} \quad x > x^* \end{array} \right\}$$

where,

$$x^* = (\exp\rho)(1 - [1 - \exp(-2\rho)]^{1/2}).$$

Hint: use a similar conceptual approach to the one that we used in class when $w = 0$.

Problem 4: Consider an optimal investment problem.

- Every period you draw a cost c (distributed uniformly between 0 and 1) for completing a project.
- If you undertake the project, you pay c, and complete the project with probability $1 - p$.
- Each period in which the project remains uncompleted, you pay a late fee of l.
- The game continues until you complete the project.

a. Write down the Bellman Equation assuming no discounting. Why is it OK to assume no discounting in this problem.

b. Derive the optimal threshold: $c^* = \sqrt{2l}$. Explain intuitively, why this threshold does not depend on the probability of failing to complete the project, p.

c. How would these results change if we added discounting to the framework? Redo steps a and b, assuming that the agent discounts the future with discount factor $0 < \delta < 1$ and assuming that $p = 0$. Show that the optimal threshold is given by.

$$c^* = \frac{\delta - 1 + \sqrt{(1-\delta)^2 + 2\delta^2 l}}{\delta}.$$

d. When $0 < \delta < 1$, is the optimal value of c^* still independent of the value of p? If not, how does c^* qualitatively vary with p? Provide an intuitive argument.

e. Prove that the expected delay until completion is given by:

$$[c^*(1-p)]^{-1} - 1.$$

Note that this calculation takes the perspective on an agent who hasn't yet observed the current period's draw of c. So there is a $c^*(1-p)$ probability that the delay is 0.