Problem Set #4 Instructor: David Laibson
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Problem 1 (Consumption Based Capital Asset Pricing Model): We will now use the Euler Equation to derive an asset pricing equation. Recall the Euler Equation:

$$u'(c_t) = E_t \delta R^i_{t+1} u'(c_{t+1})$$

We will again linearize Euler Equation, but this time we will allow for many different types of assets,

$$R^1_t, R^2_t, ..., R^i_t, ..., R^I_t$$

with stochastic returns.

$$R^i_t = \exp(r^i_t + \sigma^i \varepsilon^i_t - \left(\sigma^i\right)^2 /2),$$

where ε^i_t is stochastic and has unit variance and all other terms on the right-hand-side are fixed.

We denote the risk-free return: R^f_t. Specifically, we will assume that at time $t-1$, R^f_t is known.

We will now solve for the equilibrium relationship between asset returns and consumption innovations. You can think of this as an economy in which asset prices are endogenous and the consumption process is fixed. However, what we will derive below applies to an economy with either (or both) endogenous asset prices or endogenous consumption.

Assume u is in the class of isoelastic (i.e., constant relative risk aversion) utility functions,

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}.$$

1. Rearrange the Euler Equation to find (hint: see lecture notes from lecture 2),

$$1 = E_t \exp\left[-\rho + r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \left(\sigma^i\right)^2/2 - \gamma \Delta \ln c_{t+1}\right]$$

   where $-\ln \delta = \rho$.

2. Assume that ε^i_{t+1} and $\Delta \ln c_{t+1}$ are conditionally normally distributed. Show that

$$0 = r^i_{+1} - \left(\sigma^i\right)^2/2 - \rho - \gamma E \Delta \ln c_{+1} + \frac{1}{2} V(\sigma^i \varepsilon^i_{+1} - \gamma \Delta \ln c_{+1}). \quad (1)$$

where the expectation operator E and the variance operator V are conditional on information at date t. Set $r^i_{+1} = r^f_{+1}$, to show that

$$0 = r^f_{+1} - \rho - \gamma E\Delta \ln c_{+1} + \frac{1}{2}V(\gamma\Delta \ln c_{+1}).$$

Difference Eq. 1 for assets i and j, to find

$$r^i_{+1} - r^j_{+1} = \frac{1}{2}\left[\left(\sigma^i\right)^2 - \left(\sigma^j\right)^2 - V(\sigma^i\varepsilon^i_{+1} - \gamma\Delta \ln c_{+1}) + V(\sigma^j\varepsilon^j_{+1} - \gamma\Delta \ln c_{+1})\right].$$

3. Prove the following statistical lemma: If A and B are random variables, then,

$$V(A + B) = V(A) + V(B) + 2Cov(A, B)$$

4. Using the statistical lemma, show that

$$\pi^{ij}_{+1} = \gamma\sigma_{ic} - \gamma\sigma_{jc}$$

where $\pi^{ij} \equiv r^i - r^j$, and $\sigma_{ic} \equiv Cov(\sigma^i\varepsilon^i, \Delta \ln c)$.

5. Now consider the special case where i =equities and j =risk free asset. Show that

$$\pi^{equity,f} = \gamma\sigma_{equity,c}.$$

Interpret this equation.

Problem 2 (Asset pricing and the equity model): Reconsider the equity model from the first problem set. (Note that this is just a stochastic version of the eat-the-pie problem.) Here is the equity problem again, if you don't remember it:

Assume that a consumer with only equity wealth must choose period-by-period consumption in a discrete-time dynamic optimization problem. Specifically, consider the sequence problem:

$$v(x_0) = \sup_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \exp(-\rho t)u(c_t)$$

subject to the constraints: $x_{t+1} = \exp(r + \sigma\varepsilon_{t+1} - \sigma^2/2)(x_t - c_t) = R^{equity}_{t+1}(x_t - c_t)$, ε_t iid and $\varepsilon_t \sim N(0, 1)$, $c_t \in [0, x_t]$, $x_0 > 0$. Here x_t represents equity wealth at period t and c_t represents consumption at period t. The consumer has discount rate ρ and the consumer can only invest in a risky asset with expected return $ER^{equity} = E\exp(r + \sigma\varepsilon - \sigma^2/2) = \exp(r)$. Finally, assume that the consumer has an isoelastic utility function:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

with $\gamma \in [0, \infty]$, $\gamma \neq 1$. Note that this utility function has the property of constant relative risk aversion

$$\frac{-cu''(c)}{u'(c)} = \gamma.$$

This scaling property enables us to analytically solve this problem.

The associated Bellman equation for this problem is given by

$$v(x) = \sup_{y \in [0,x]} u(x-y) + E\exp(-\rho)v(\exp(r + \sigma\varepsilon - \sigma^2/2)y).$$

The value function takes the special form

$$v(x) = \phi\frac{x^{1-\gamma}}{1-\gamma}.$$

We previously showed that the consumption function takes the form,

$$c = \phi^{-\frac{1}{\gamma}}x.$$

where,

$$\ln(1 - \phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma}\left[(1-\gamma)r - \rho\right] + \frac{1}{2}(\gamma - 1)\sigma^2$$

We also showed that,

$$E\ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\gamma}(r - \rho) + \frac{\gamma}{2}\sigma^2 - \sigma^2.$$

We are now going to reconsider the issues posed at the end of the first problem set. In particular, we are going to use the Euler Equation to price a risk free bond that is in zero net supply (i.e., every long position in this bond must be offset by a short position). We will ask what interest rate would make a consumer just willing to hold an arbitrarily small amount of the risk free bond. In particular, we will assume that the amount is sufficiently small that it does not effect the properties of the consumption process. Hence, we can use the stochastic dynamics of the consumption process, which were derived above, to price the bond.

1. Note that the Euler Equation must hold for *all assets* in the consumer's portfolio. Explain intuitively why this is the case.

2. If the risk free bond has interest rate R^f (with $\ln R^f = r^f$), show that the Euler Equation for the risk free asset will be,

$$u'(c_t) = E_t R^f \exp(-\rho) u'(c_{t+1}).$$

3. Manipulate the Euler Equation to show that,

$$\left(R^f\right)^{-1} = E_t \exp\left\{-\rho - \gamma \Delta \ln c_{t+1}\right\}.$$

4. Show that $\Delta \ln c_{t+1}$ is distributed *normally* with mean $\frac{1}{\gamma}(r-\rho)+\frac{\gamma}{2}\sigma^2-\sigma^2$ and variance σ^2. Note that we usually just assume that $\Delta \ln c_{t+1}$ is distributed normally. For this problem, we can show it exactly. You can use the intermediate results derived on the first problem set (which are summarized above).

5. Use the result of questions 3 and 4 to derive the equilibrium interest rate of the risk free bond. Now use the Consumption Capital Asset Pricing Equation (previous problem) to derive the equilibrium interest rate of the risk free bond. Your results should be the same, since both derivations are based on the Euler Equation.

6. Defend the assumption that the amount of risk free bond is in zero net supply (i.e., that the net amount available is zero).

7. In this model economy $\sigma_{equity,c} = \sigma^2_{equity}$. Is this true in the real world? Why or why not?

Problem 3: [48 minute question on asset pricing from the General – I've left it unchanged so you see what an exam question can look like] This question has many parts. If you get stuck on one part, feel free to skip it and continue with the rest. Please write clearly, as your answers will be judged mostly on the quality of your argument (and not just your final mathematical formulae).

- Consider a "static" economy that exists for only two periods. In period 0, agents trade claims (and do *not* consume anything). In period 1 consumption takes place.
- The economy has two assets that pay out in period 1. (Note that neither asset has continuation value after period 1, since the economy ends at the end of period 1.) The two assets are:
  - A tree that pays out a sure-thing of c in period 1.
  - And a firm with stochastic dividends. With probability $0 < \mu < 1$ the firm generates 1 unit of dividends in period 1. With probability $1 - \mu$ the firm generates 0 dividends in period 1.
- Assume that there is a representative agent with ln (natural log) utility.
- Assume that the economy is in competitive equilibrium.
- Assume that consumption (in period 1) is the numeraire. We will express the equilibrium price of the firm at date zero as p units of consumption (to be paid in period 1). In other words, one can buy Δ fraction of the firm during the trading period (i.e., during period 0) by promising to give the seller a claim that is worth $\Delta \cdot p$ units of consumption in period 1.

1. Use a perturbation argument to explain why the following Euler Equation characterizes the equilibrium price at date zero, p.

$$(1-p)\,\mu\, u'(c+1) - p\,(1-\mu)\,u'(c) = 0$$

2. Show that

$$\lim_{c \to \infty} p = \mu.$$

In other words, show that the expected gross "return" on the firm is

$$\lim_{c \to \infty} E_0 R \equiv \lim_{c \to \infty} \frac{E_0(\text{payout})}{p} = 1$$

3. Explain the results in part 2 using economic intuition. Specifically, explain why the equilibrium price of the firm is equal to the expected value of the firm (when c is large). Explain why risk aversion does not play a role here, even though the firm has very volatile dividends. (Hint: the data-generating processes are not log-normal, so you need to be selective in the arguments that you use.)

4. Again consider the same setting, but *now* assume that the tree is also risky. This new assumption will be in force for the rest of the problem. Specifically, assume that there are now two states. In the good state, the tree pays $(1 + \alpha)c$ and the firm pays 1 (assume $\alpha > 0$). In the bad state the tree pays c and the firm pays nothing. As before, the good state occurs with probability μ and the bad state occurs with probability $1 - \mu$.

5. Use a perturbation argument to explain why the following (slightly different) Euler Equation characterizes the equilibrium price, p.

$$(1 - p)\, \mu\, u'((1 + \alpha)c + 1) - p\; (1 - \mu)\; u'(c) = 0$$

6. Show that

$$\lim_{c \to \infty} p = \frac{\mu}{1 + \alpha(1 - \mu)} < \mu = \text{Expected dividend}$$

   In other words, show that the expected gross "return" on the firm is

$$\lim_{c \to \infty} E_0 R \equiv \lim_{c \to \infty} \frac{E_0(\text{payout})}{p} = 1 + \alpha(1 - \mu) > 1.$$

7. Explain the results in part 6 using economic intuition. Specifically, explain why the equilbrium price of the firm is less than the expected value of the firm (whether or not c is large). Why does risk aversion now play a role? Contrast this case with the earlier case (when the tree was not risky).

8. Using the equations in part 6, one can generate a high required return on the firm by setting $\alpha = 0.12$ and $\mu = 1/2$. This calibration has the desirable property that it explains the historical equity premium of 0.06. What is wrong with such a calibration? In other words, why is such a calibration inconsistent with other empirical facts?

9. Why is the equity premium *negative* if we instead made $\alpha < 0$?