**Problem 1 (Q-theory with depreciation and taxes):** Consider an N-firm industry (N large) in which firms have instantaneous earnings flows:

$$\pi(K(t)) \cdot k(t)$$

where $K(t)$ is industry-wide capital, $k(t)$ is firm-specific capital, and $\pi'(\cdot) < 0$. The firm level capital stock adjusts according to

$$dk = (I - \gamma k)dt$$

where $I$ is the instantaneous investment flow and $\gamma$ is the depreciation rate. The price of capital is unity and firms pay convex adjustment costs $C(I(t))$, with $C(0) = 0$, $C'(0) = 0$, and $C'' > 0$. Let $q(t)$ represent the NPV of the stream of profits from a marginal unit of installed capital.

$$q(t) = \int_{s=t}^{\infty} e^{-(r+\gamma)(s-t)}\pi(K(s))ds$$

**a.** Interpret $q(t)$. How and why does this definition of $q(t)$ differ from the definition we used in class? Using the Envelope Theorem, show that $\frac{\partial V(k,K)}{\partial k} = q$, where $V(k,K)$ is the value function of a firm.

**b.** Derive the firm's Bellman Equation, and the first-order condition:

$$q(t) = 1 + C'(I(t))$$

**c.** Derive the Bellman Equation associated with $q(t)$:

$$(r + \gamma)q(t) = \pi(K(t)) + \dot{q}(t)$$

**d.** Analyze the phase diagram associated with this industry. Plot the $\dot{K} = 0$ and $\dot{q} = 0$ loci. Graph the saddle path. How does this diagram differ from the one we derived in class? How does a rise in the depreciation rate affect the steady state level of capital? Explain.

**e.** Now suppose the industry is in steady state. At time period $t_0$ the government announces a temporary investment tax credit which will last until time period $t_1$. The investment tax credit provides a $\theta$ $(-\theta)$ subsidy per unit of positive (negative) investment $(0 < \theta < 1)$. Derive the first-order condition of the firm during the transition interval $[t_0, t_1]$:

$$q(t) = 1 - \theta + C'(I(t)).$$

**f.** Plot the transition path of the industry on the phase diagram. In addition, plot the path of $K$ and $I$ against time. Explain why investment is non-monotonic over the interval $[t_0, t_1]$. Does a temporary or a permanent investment tax credit have a larger impact on short-run investment? Why?

**g.** Now suppose that the subsidy is pre-announced at period $t_{-1} < t_0$. In other words, the subsidy begins at period $t_0$, and economic agents find out about it at time $t_{-1}$. Plot the transition path of the industry on the phase diagram. In addition, plot the path of $K$ and $I$ against time.

**Problem 2 (True, False, or Uncertain).** Please explain whether the following statements are True, False, or Uncertain. You will be graded on the quality of your *explanation.*

**a.** Ito's Lemma implies that second-order terms in the total differential of the value function vanish as $dt \rightarrow 0$.

**b.** If an investment tax credit is anticipated before the tax credit starts, investment will rise above its steady state level before the tax credit starts. When the tax credit starts, investment will jump up even more.

**Problem 3 (Bertola and Caballero 1990):** Consider a firm that faces a decision problem. The firm tries to keep an Ito process, $X(t)$, near zero. Specifically, the firm tries to maximize the net present value of instantaneous payoffs

$$-\frac{b}{2}X^2$$

where the discount rate is $\rho$. Assume $X(t)$ is described by

$$dX = \alpha dt + \sigma dz + (\text{discrete adjustments undertaken by the firm}).$$

A discrete upward adjustment of $I$ units costs the firm $C_u + c_u I$, with $C_u \geq 0$ and $c_u \geq 0$. A discrete downward adjustment of $|I|$ units costs the firm $C_d + c_d|I|$, with $C_d \geq 0$ and $c_d \geq 0$. Solving this problem, is equivalent to finding four endogenous boundaries for $X(t)$: $U$, $u$, $d$, $D$. When $X(t)$ reaches $U$, the firm discretely raises $X(t)$, jumping to $X(t) = u$. When $X(t)$ reaches $D$, the firm discretely lowers $X(t)$, jumping to $X(t) = d$.

**a.** Economically motivate this problem. Specifically, find a convincing economic problem that is well-described by the assumptions summarized above.

**b.** Intuitively explain why $u \leq d$. Under what conditions will this inequality hold strictly? When will $u = d$. Explain your reasoning with intuition. Intuitively explain why $U \leq u$ and $d \leq D$. Under what conditions will these inequalities hold strictly? When will $u = U$? When will $d = D$? Explain your reasoning with intuition.

**c.** Derive the continuous-time Bellman Equation in the continuation region (i.e., for $U \leq X(t) \leq D$). Apply Ito's Lemma to show that in the continuation region

$$\rho V(X) = -\frac{b}{2}X^2 + \alpha V'(X) + \frac{1}{2}\sigma^2 V''(X) \tag{1}$$

Why does $\frac{\partial V}{\partial t}$ not appear in this equation? Interpret this equation as a decomposition of the required return.

**d.** Show that

$$V(X) = -\frac{b}{2}\left(\frac{X^2}{\rho} + \frac{\sigma^2 + 2\alpha X}{\rho^2} + \frac{2\alpha^2}{\rho^3}\right)$$

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is a solution to this equation. Show that this is the expected present value of the firm's payoff stream assuming that adjustment costs are infinite.

**e.** Show that the general solution to Equation 1 is given by

$$V(X) = -\frac{b}{2}\left(\frac{X^2}{\rho} + \frac{\sigma^2 + 2\alpha X}{\rho^2} + \frac{2\alpha^2}{\rho^3}\right) + A_1 e^{\alpha_1 X} + A_2 e^{\alpha_2 X}$$

with roots

$$\alpha_1 = \frac{-\alpha + \sqrt{\alpha^2 + 2\sigma^2 \rho}}{\sigma^2}$$

$$\alpha_2 = \frac{-\alpha - \sqrt{\alpha^2 + 2\sigma^2 \rho}}{\sigma^2}.$$

Sign roots $\alpha_1$ and $\alpha_2$.

**f.** Write down the six equation system that you would use to solve for the unknown variables that characterize the firm's value function. Intuitively explain all of the equations. Why do we need six equations? (You do not actually need to solve the system.)

**g.** Use intuition to explain how the threshold values $U$ and $D$ change with $\sigma$.

**h.** Is it possible to parameterize the model in a way that makes $U$ greater than 0? $D$ less than 0? $u$ greater than 0? $d$ less than 0? Explain intuitively.